The Mighty, Mighty Logarithm

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Logarithms: some history

- The idea was invented by John Napier (1550-1617)
  (About the time of Shakespeare (1564-1616))
- John Napier is famous for testing his servants for theft using a black rooster. He covered it in soot and placed it in a darkened room. He then told the servants that the rooster could psychically tell if they were a thief by touch. He then made his servants to go in one by one and pet him. You can guess the rest...
- Invented (1614) Logs to help him with his calculations
- Johannes Kepler, used it for planetary orbit calculations, and from then it caught on

Logarithms: the name

- Napier also coined the term *Logarithmus* (in Latin)
- Logarithmus = "ratio-number,"
  - from Greek logos "ratio" + arithmos "number"
  - though perhaps he used "logos" in the sense of "calculation"
- Henry Briggs first used the English word Logarithm

http://jeff560.tripod.com/l.html

Logs: a definition

The "log" function is the inverse of the exponential, for instance, if

$$x = 10^y$$

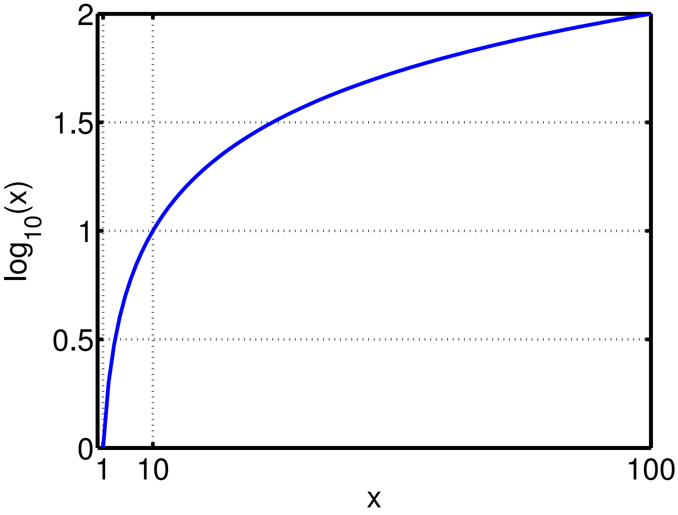
then we can reverse the equation as follows

$$y = log_{10} x$$

for example:

$$\log_{10} 100 = \log_{10} \left(10^2\right) = 2$$

Logs: a graph



Logs: a definition

We can do logs to any base, i.e., base a

$$x = a^y$$

then we can reverse the equation as follows

$$y = log_a x$$

for example:

$$\log_2 8 = \log_2 \left(2^3\right) = 3$$

But from now on, I will just write log when it doesn't matter.

Logs: the key property

$$\log(xy) = \log(x) + \log(y)$$

Logs: we can use that to multiply big numbers

$$xy = a^{\log_a(x)+\log_a(y)}$$

So we just

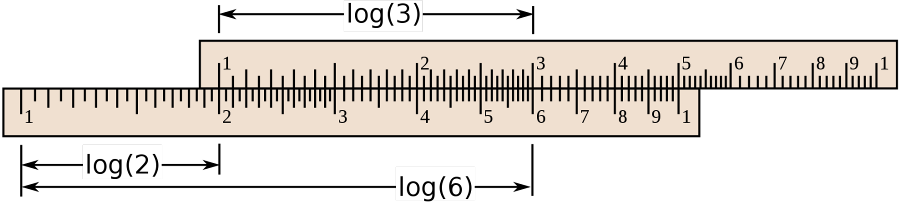
- take the logs of the two numbers
- add the logs together
- take them to the power of (in this case) a

We can also do division, calculate square roots, and do many other calculations much more easily this way.

Logs: tables

- For hundreds of years, mathematicians and engineeers used logs to perform complex calculations
- Calculating the logs themselves was hard though
- So people wrote out, and printed, entire books of tables of logs
- Eventually the tables were replaced by the slide rule
    - much faster than tables
    - but somewhat less accurate (used by engineers)
- Eventually all of that was replace by the pocket calculator
    - but lots of calculators can still calculate logs

Logs: slide rules rule



http://en.wikipedia.org/wiki/File:Slide_rule_example2_with_labels.svg

Logarithms: activity

Make your own slide rule

http://www.csiro.au/helix/mathsbyemail/activity/sliderule.html

- Line up the 1 on the first ruler with the 2 on the second
- Find the 3 on the first ruler
- Look at the number it lines up with on the second

Weber-Fechner law

- Really two laws:
  - Weber's law – just noticeable difference between two stimuli is proportional to the magnitude of the stimuli
  - Fechner's law states that subjective sensation is proportional to the logarithm of the stimulus intensity.
- You need to study differential equations to see that these are the same thing – maybe later when you are Uni.
- You can see them in the way we measure stimuli

http://en.wikipedia.org/wiki/Weber-Fechner_law

Weber-Fechner law: sound

- We measure sound levels using the deci-Bell (or dB) scale
- dB scale

$$measurement = 10 \log_{10} \left(\frac{power}{10^{-12}} \right) dB$$

- the deci- corresponds to the extra factor of 10 at the front
- So
  - 10 dB corresponds to a factor of 10 in power

dB

Example	Sound Pressure Level (dB)	Sound Intensity (watts/m^2)
Snare drums, played hard at 6 inches	150	1000
30m from jet aircraft	140	100
Threshold of pain	130	10
Jack hammer	120	1
Fender guitar amplifier, full volume at 10 inches	110	0.1
Subway	100	0.01
	90	0.001
Typical home stereo listening level	80	0.0001
Kerbside of busy road	70	0.00001
Conversational speech at 1 foot away	60	10^{-6}
Average office noise	50	10^{-7}
Quiet conversation	40	10^{-8}
Quiet office	30	10^{-9}
Quiet living room	20	10^{-10}
Quiet recording studio	10	10^{-11}
Threshold of hearing for healthy youths	0	10^{-12}

Weber-Fechner law: sight

- We measure stellar magnitude

$$magnitude - m_0 = -2.5 \log_{10} \left(\frac{F}{F_0} \right)$$

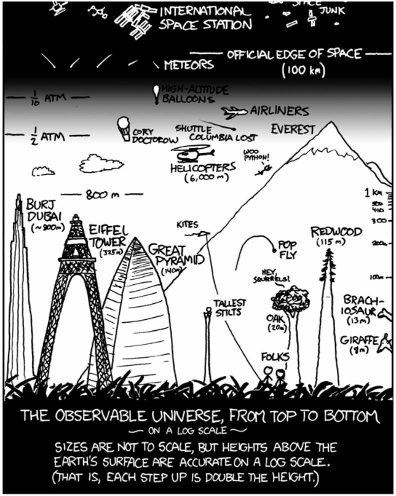
  - F is observed flux
  - m_0 and F_0 are reference magnitudes and flux
  - invented by Hipparchus in 150 B.C. (before we formally knew about logs)
  - notice its a negative scale
    - brighter stars have lower magnitudes
    - may be measured per frequency band

Weber-Fechner law

Actually, its not really this simple,
http://en.wikipedia.org/wiki/Stevens'_power_law
but there are lots of other cases:

- dB is used in lots of electronics (e.g. radar)
- music scale (octaves)
- weight perception
- Perception of time
- Perception of the value of money
- pH scale for acidity/alkalinity
- Earthquakes – the scale we use to measure them is the Richter scale
  - at 3, you might only just notice and earthquake (like 480 kg explosion)
  - at 6, buildings would be badly damaged (like a 15 kiloton explosion)
  - at 9, death toll would be in thousands to millions (like a 480 megaton explosion)

Log plots



http://xkcd.com/482/

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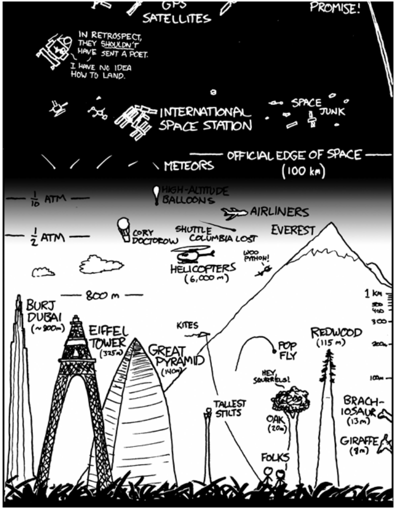

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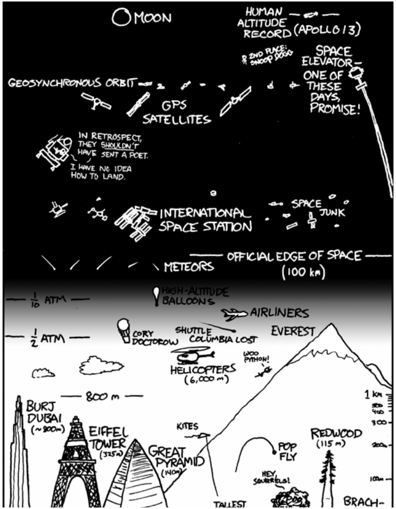

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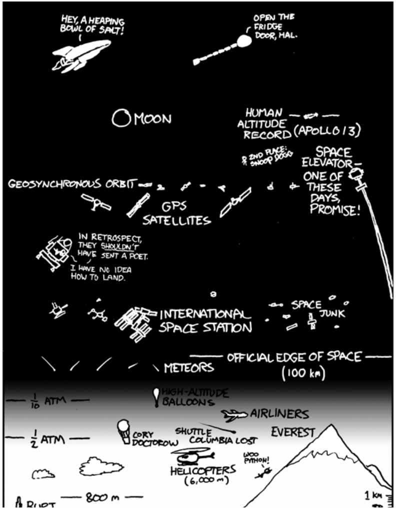

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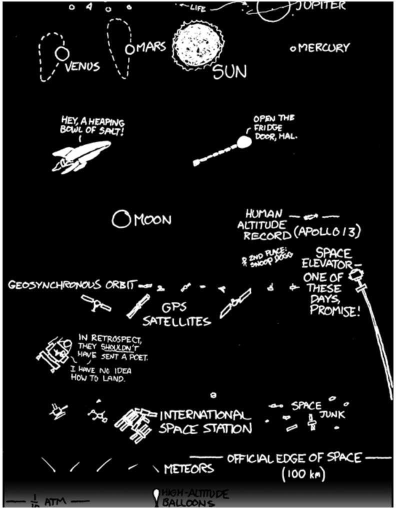

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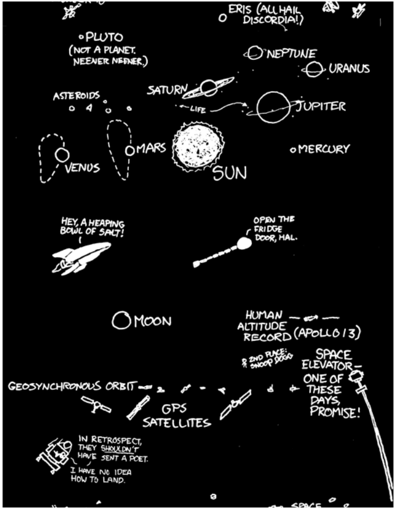

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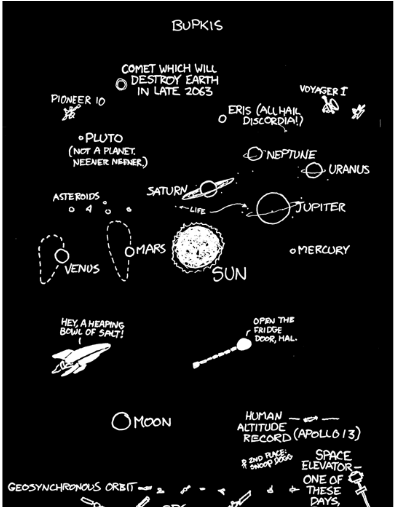

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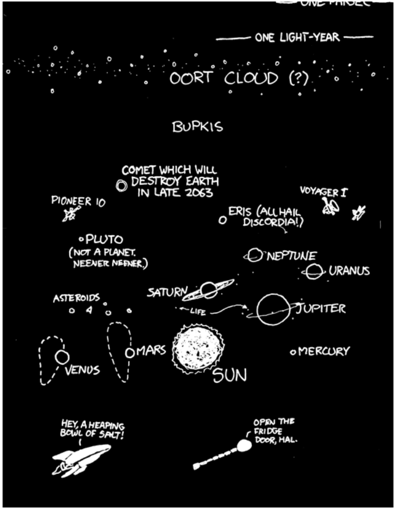

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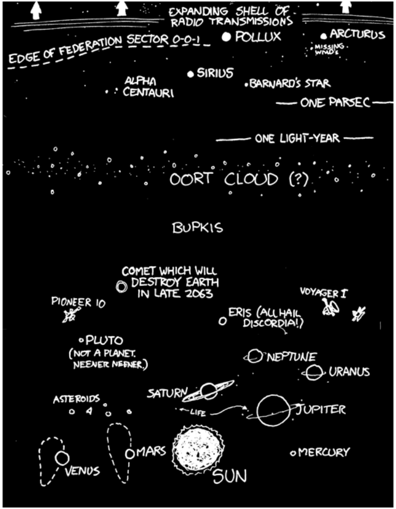

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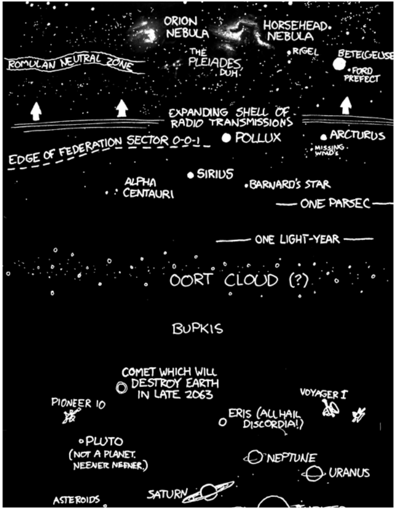

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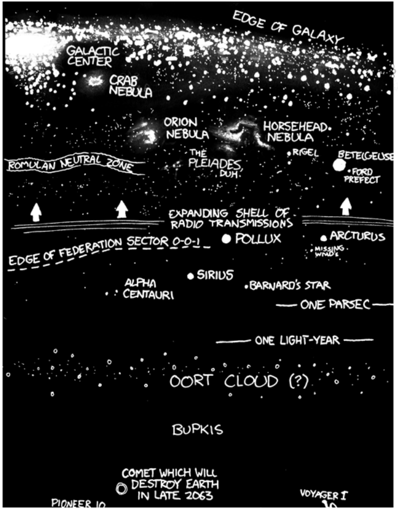

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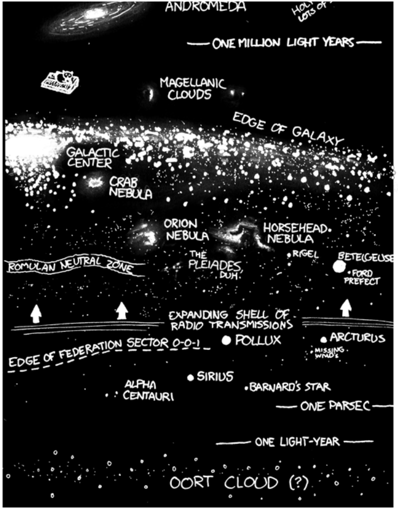

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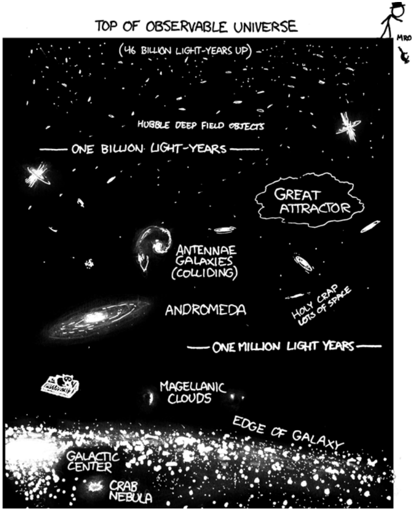

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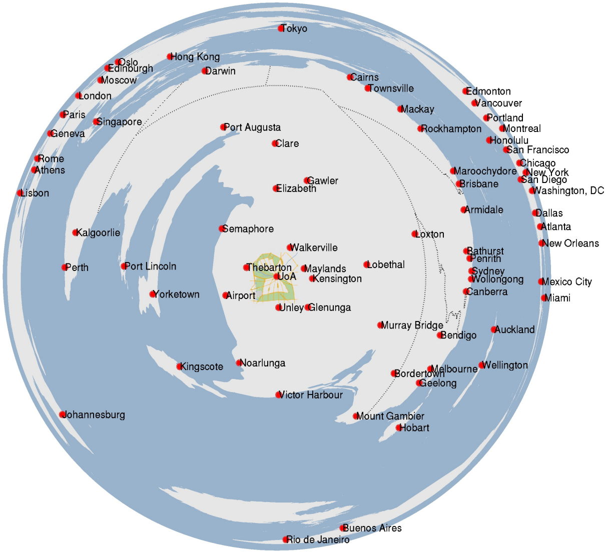
Log plots

- Let you compare highly variable data
  - really big things with really small things
  - you can see them on the same scale
  - big things don't dwarf the small things

Log-azimuthal map



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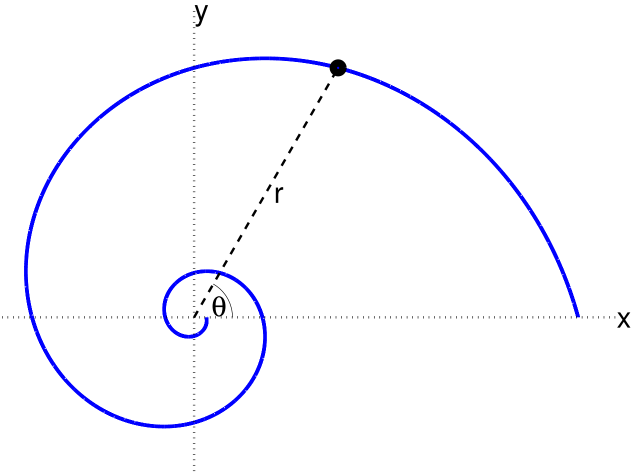
Logs in nature: the log spiral

Let's create a spiral using this function:

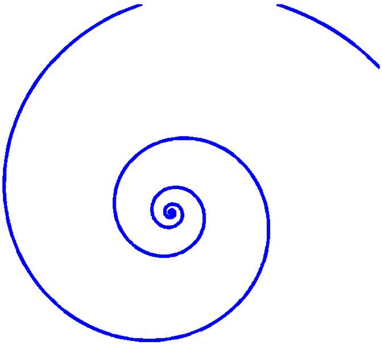
$$r = ae^{b\theta} \qquad \text{or} \qquad \theta = \frac{1}{b}\log_e\left(\frac{r}{a}\right)$$



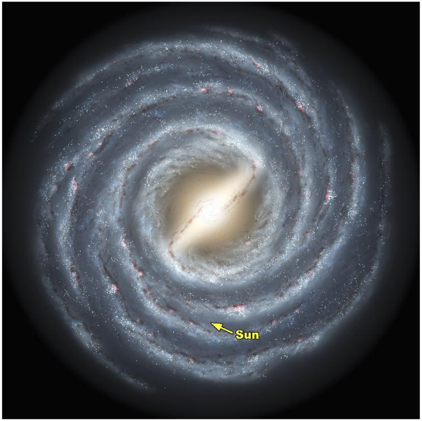
Logs in nature: the log spiral

- We call $\log_e$ the natural log and write it ln
- Jakob Bernoulli called the curve spira mirabilis (marvelous spiral) because it has lots of interesting properties:
    - its also called the equiangular spiral because the angle between a tangent, and the radial line is fixed (as we will see later)
    - it has a bunch of other nice mathematical properties http://jwilson.coe.uga.edu/EMT668/EMAT6680.F99/Erbas/KURSATgeometrypro/relatedcurves/relatedcurves.html
- Its related to
    - Fiboacci sequence
    - the Golden ratio ϕ
- It is "self-similar"
- We often see it in nature

Self-similar spirals

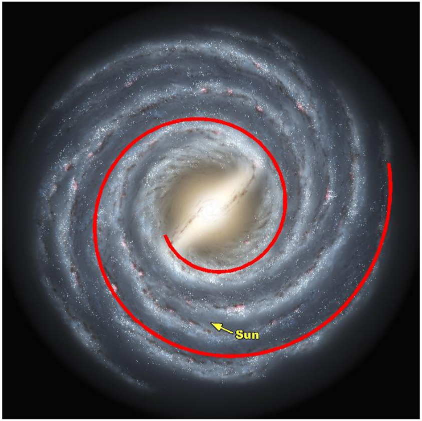


Log spirals: e.g., spiral galaxies



Milky Way (our galaxy) from
http://andromida.hubpages.com/hub/milky-way-galaxy

Log spirals: e.g., spiral galaxies



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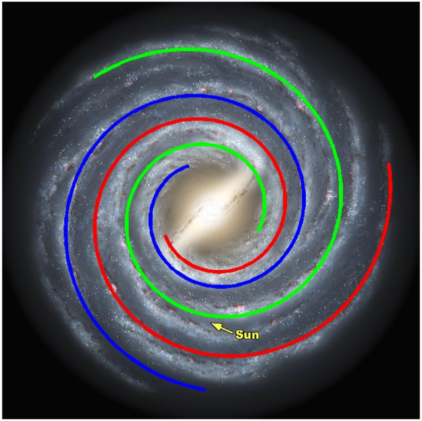

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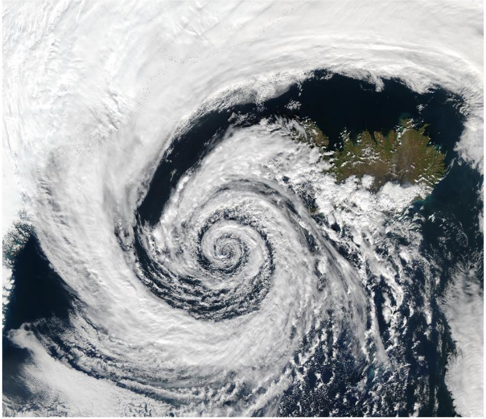
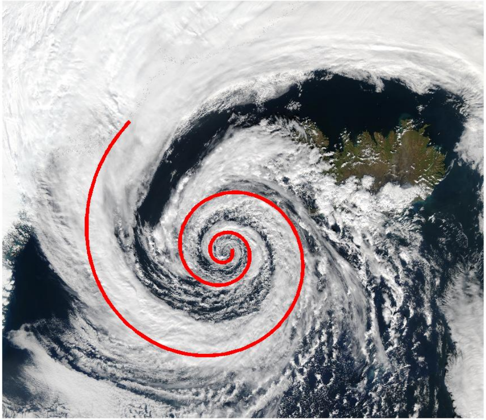
Log spirals: e.g., storms



Storm over Iceland from
http://en.wikipedia.org/wiki/File:Low_pressure_system_over_Iceland.jpg

Log spirals: e.g., storms



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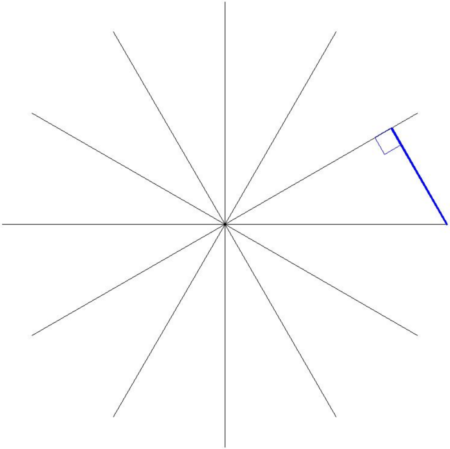
How to draw a log spiral

Start a point on a spoke and draw a line at right angles to the next spoke.



Then keep going inwards.

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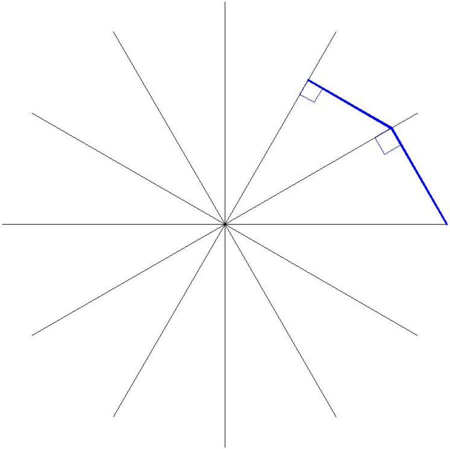

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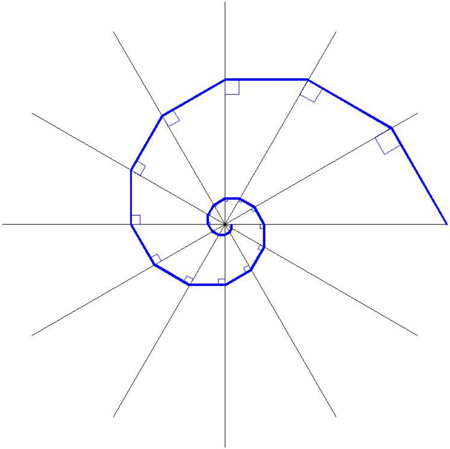

The more spokes you have the more accurate the spiral.

How to draw a log spiral

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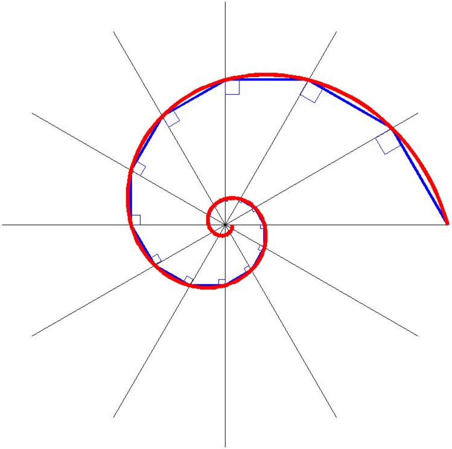
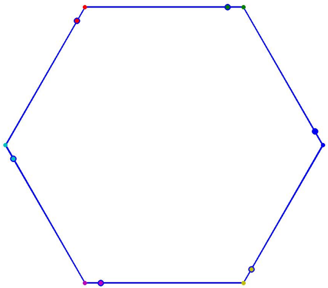


The more spokes you have the more accurate the spiral.

Pursuit curves

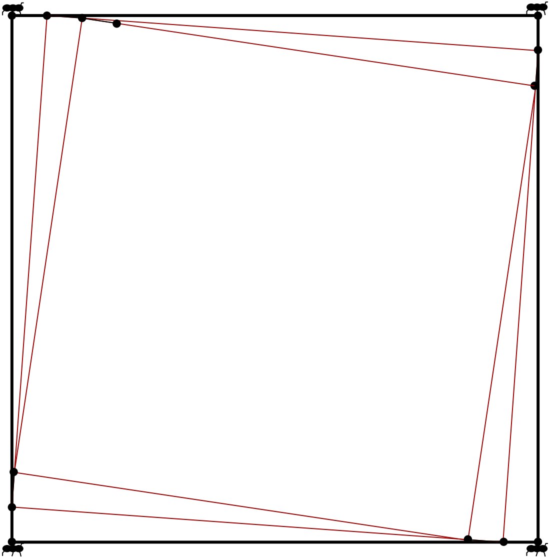
- imagine an ant starting at each corner
- he pursues the ant clockwise from himself
- always steers directly towards that ant



http://jwilson.coe.uga.edu/EMT668/EMAT6680.F99/Erbas/KURSATgeometrypro/relatedcurves/relatedcurves.html

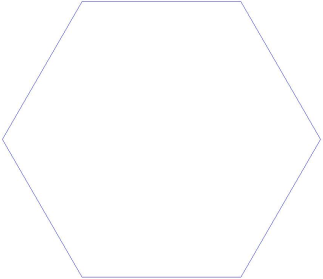
Pursuit curves

Each ant pursues the one clockwise from himself.



Pursuit curves

- you also get a nice pattern of lines



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Shells

Nautilus shell

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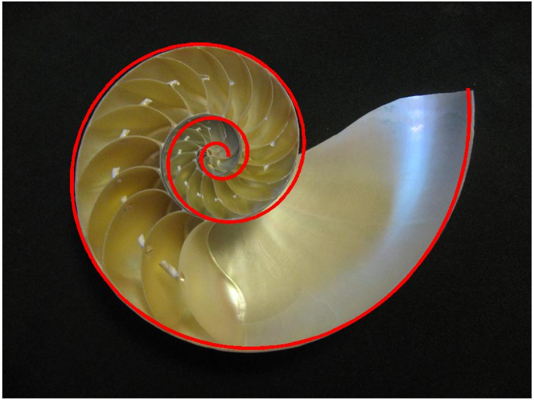

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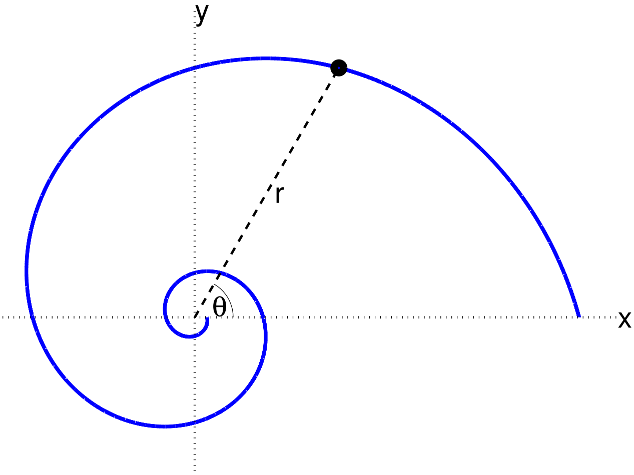
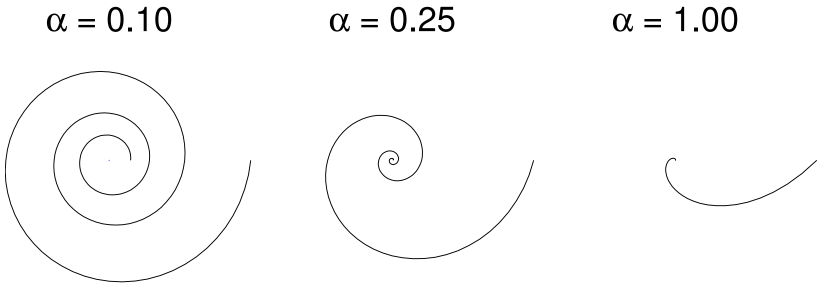


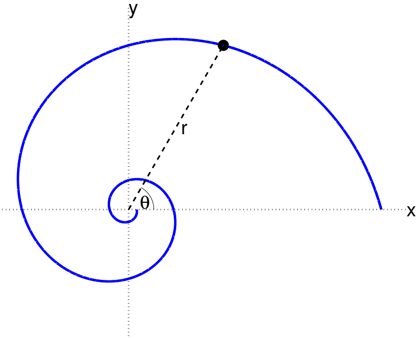
Parameters

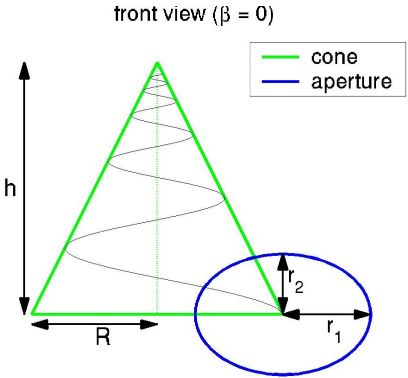
- b is just an arbitrary starting point
- a determines how tight the spiral is
- a also determines the direction
  - $a > 0$ then anticlockwise (as you move inwards)
  - $a < 0$ then clockwise (as you move inwards)



Shells

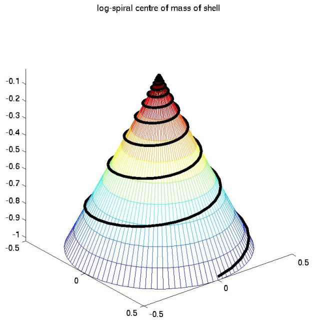
- View from the top is a log spiral
- View from the side is a cone

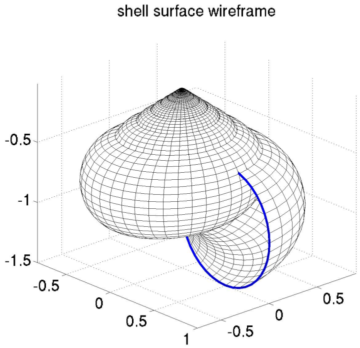




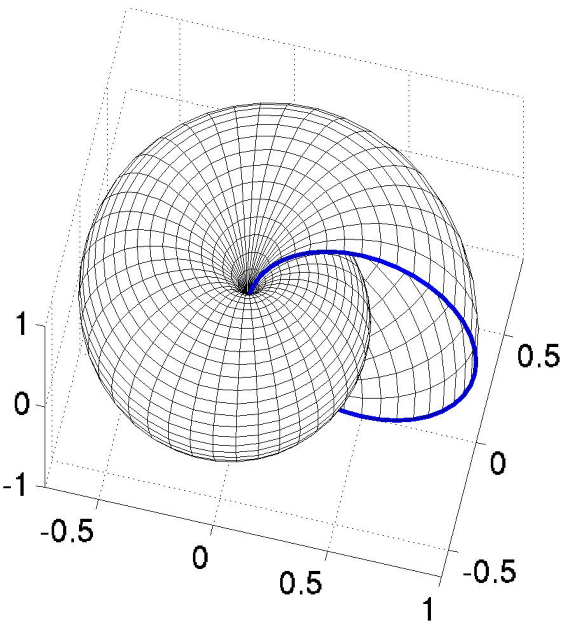
Shells

- Now rotate an elipse (or circle) around this curve





shell surface wireframe



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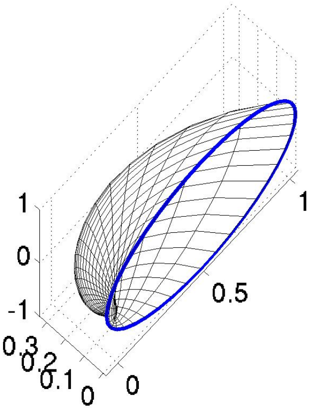

Shell generator

Login:

```
Username:  megamaths
Password:  Maths*5
```

- WWW shell generator:
  http://bandicoot.maths.adelaide.edu.au/shells/shell.cgi
  Login and start up a web browser. Point it at this URL and fill in the parameters.
- Matlab code:
  http://www.maths.adelaide.edu.au/matthew.roughan/maths_talks.html
  Login and start Matlab. Set the parameters (see the sheet), and then call the 'shell' to generate some pictures.

Other uses of logarithms

- Entropy
- Calculating computational complexity
- Music
- Number theory
- Hick's law http://en.wikipedia.org/wiki/Hick's_law
- Fitt's law http://en.wikipedia.org/wiki/Fitts's_law