Variational Methods & Optimal Control

lecture 22

Matthew Roughan
<matthew.roughan@adelaide.edu.au>

Discipline of Applied Mathematics
School of Mathematical Sciences
University of Adelaide

April 14, 2016

More Optimal Control Examples

First we'll cover a bit more terminology, and then some examples primarily focussed on planned growth strategies in economics.

Formulation of control problems

We break a control problem into two parts

- **The system state:** $\mathbf{x}(t) = (x_1(t), \ldots, x_n(t))^t$
  The system state describes the system (e.g. position and velocity of the car in car parking example)
- **The control:** $\mathbf{u}(t) = (u_1(t), \ldots, u_m(t))^t$
  We apply the control to the system (e.g. force applied to the car).

The evolution of the system is governed by the set of DEs

$$\dot{\mathbf{x}}(t) = \mathbf{g}(t, \mathbf{x}, \mathbf{u})$$

In a control problem we want to get the system to a particular state $\mathbf{x}(t_1)$ at time t_1, given initial state $\mathbf{x}(t_0)$.

Optimal control problems

In an **optimal** control problem we have still have the system equations $\dot{\mathbf{x}}(t) = \mathbf{g}(t, \mathbf{x}, \mathbf{u})$ and we might wish to get to state $\mathbf{x}(t_1)$ given initial state $\mathbf{x}(t_0)$, but now we wish to do so while minimizing a functional

$$F\{\mathbf{x}, \mathbf{u}\} = \int_{t_0}^{t_1} f(t, \mathbf{x}, \mathbf{u})\, dt$$

That is, we wish to choose a function $\mathbf{u}(t)$ which minimizes the functional $F\{\mathbf{x}, \mathbf{u}\}$, while satisfying the end-point conditions $\mathbf{x}(t_0) = \mathbf{x}_0$ and $\mathbf{x}(t_1) = \mathbf{x}_1$, and the non-holonomic constraints $\dot{\mathbf{x}}(t) = \mathbf{g}(t, \mathbf{x}, \mathbf{u})$.

Optimal control problems

Optimization functional

$$F\{\mathbf{x},\mathbf{u}\} = \int_{t_0}^{t_1} f(t,\mathbf{x},\mathbf{u})\,dt$$

Note that

- $f(t,\mathbf{x},\mathbf{u})$ has no dependence on $\dot{\mathbf{u}}$: this is typically because costs depend on the control, not how we change the control, but there might be counter-examples
- $f(t,\mathbf{x},\mathbf{u})$ has no dependence on $\dot{\mathbf{x}}$: this is common in control problems, but not universal (we have seen at least one counter example).

Terminal costs

Sometimes in optimal control we don't fix the end-point $\mathbf{x}(t_1)$, but rather we assign a cost $\phi(t_1,\mathbf{x}(t_1))$ to particular end-points.

So now we wish to choose a control $\mathbf{u}(t)$ which minimizes the functional

$$F\{\mathbf{x},\mathbf{u}\} = \phi(t_1,\mathbf{x}(t_1)) + \int_{t_0}^{t_1} f(t,\mathbf{x},\mathbf{u})\,dt$$

while satisfying the single end-point condition $\mathbf{x}(t_0)=\mathbf{x}_0$, and the non-holonomic constraint $\dot{\mathbf{x}}(t)=\mathbf{g}(t,\mathbf{x},\mathbf{u})$.

- $\phi(t_1,\mathbf{x}(t_1))$ is called the **terminal cost**.

System Terminology

- **linear:** the state equations are a set of linear DEs.
- **autonomous:** time doesn't appear explicitly in the state equations (e.g. in $g(\mathbf{x},\mathbf{u})$, or $f(\mathbf{x},\mathbf{u})$).
  - also called time-invariant
- **terminal cost:** the term $\phi(t_1,\mathbf{x}(t_1))$ is called the terminal cost.
- **controllable:** a solution to the control problem exists.
- **stable:** a stable equilibrium solution to the system DEs exists.
  - often we are interested in problems that are unstable, or we wouldn't really need a control

Control Terminology

- control (driver or automatic)
  - **planned** (open loop)
  - **feedback** (closed loop) control depends on current state
- type of control
  - movement from A to B
  - continuous operations (maintain equilibrium)
- type of cost functional F
  - minimum time
  - minimum fuel
  - quadratic costs
- admissible controls
  - unbounded/bounded/bang-bang

Cost functional examples

- **minimum time:** choose the fastest possible control

$$F\{x,u\} = \int_{t_0}^{t_1} dt$$

- **minimum fuel:** fuel is expended by the controller, and we wish to minimize this

$$F\{x,u\} = \int_{t_0}^{t_1} |u(t)|\, dt$$

- **quadratic costs:**

$$F\{x,u\} = \int_{t_0}^{t_1} x^2(t) + \alpha u^2(t)\, dt$$

Boundary conditions

- End time t_1: can be fixed or free
- End position $\mathbf{x}(t_1)$: can be fixed or free

In the cases with free boundary conditions, we introduce natural, or transversal boundary conditions.

Example: dynamic production

- A producer in purely competitive market
  - A large numbers of independent producers
  - Standardized product, e.g. potatoes
  - Firms are ”price takers”, i.e. they have no significant control over product price
  - Free entry and exit
  - Free flow of information
- wants to find optimal production path $x(t)$, $0 \leq t \leq T$.
- production target $x(T) = x_T$
- profit at time t is $\pi(x, \dot{x}, t)$
- maximize profit functional $F\{x\} = \int_0^T \pi(x, \dot{x}, t)\, dt$

Example: dynamic production

Profit calculation

- quadratic production costs $C_1 = a_1 x^2 + b_1 x + c_1$
  - labor
  - raw materials
- production increase costs $C_2 = a_2 \dot{x}^2 + b_2 \dot{x} + c_2$
  - new buildings
  - recruiting and training costs
- revenue $r = px$ where p is the constant price per unit
  - $p = const$ due to purely competitive market
- profit at time t is

$$\pi(x, \dot{x}, t) = px - C_1(x) - C_2(\dot{x})$$

Example: dynamic production

Problem formulation: maximize total profit

$$F\{x\} = \int_0^T px - C_1(x) - C_2(\dot{x})\, dt$$

subject to $x(0) = 0$ and $x(T) = x_T$.

- notice that the control, and rate of change of state are the same (i.e., $u = \dot{x}$) but we write it as above for simplicity
- autonomous problem
- the control is planned, and has quadratic costs
- admissible controls are unbounded

Example: dynamic production

Euler-Lagrange equations

$$\begin{aligned}
\frac{d}{dt}\frac{\partial \pi}{\partial \dot{x}} - \frac{\partial \pi}{\partial x} &= 0 \\
-\frac{d}{dt}\frac{\partial C_2}{\partial \dot{x}} - p + \frac{\partial C_1}{\partial x} &= 0 \\
-\frac{d}{dt}\left[2a_2\dot{x} + b_2\right] - p + 2a_1 x + b_1 &= 0 \\
-2a_2\ddot{x} - p + 2a_1 x + b_1 &= 0 \\
\ddot{x} - \frac{a_1}{a_2}x &= \frac{-p + b_1}{2a_2}
\end{aligned}$$

for $a_2 \neq 0$

Example: dynamic production

Solution (for $a_1, a_2 \neq 0$)

$$x(t) = Ae^{\sqrt{\frac{a_1}{a_2}}t} + Be^{-\sqrt{\frac{a_1}{a_2}}t} + \frac{b_1 - p}{2a_2}$$

where A and B are determined by the fixed end points $x(0) = x_0$ and $x(T) = X_T$.

This gives the optimal production schedule.

- no dependence on c_1 or c_2 (these are constant costs and so shouldn't effect production strategy)
- no dependence on b_2 because this is a linear cost in increasing production, and so occurs regardless of how we increase over time (to get to the final production target $x(T) = X_T$).

Example: dynamic production

What happens if we make the end point $x(T)$ free, i.e. we don't have a production target at time T?

Then we get a natural boundary condition

$$\left.\frac{\partial \pi}{\partial \dot{x}}\right|_{t=T} = \left.\frac{\partial C_2}{\partial \dot{x}}\right|_{t=T} = 2a_2\dot{x} + b_2\big|_{t=T} = 0$$

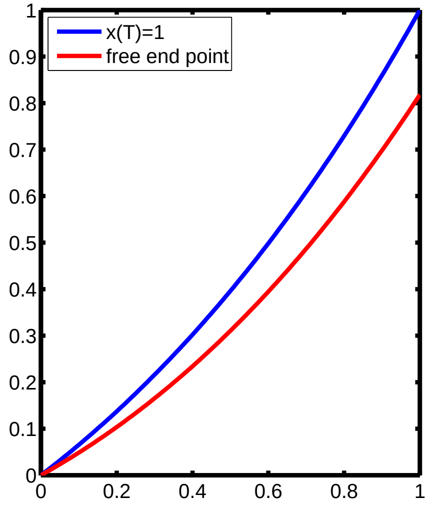
So, rearranging, we get

$$\dot{x}(T) = -\frac{b_2}{2a_2}$$

- constants A and B are determined by end-point conditions $x(0) = x_0$ and $\dot{x}(T) = -\frac{b_2}{2a_2}$

Example: dynamic production

- production costs $C_1 = x^2 + 5x$
- production increase costs $C_2 = 2\dot{x}^2 + 5\dot{x}$
- $p = 10$
- $T = 1$
- $x_0 = 0$, $x_T = 1$



Example: optimal economic growth

How much should be consumed, and how much invested for future consumption?

- optimal theory of saving (Ramsey, 1928)
- Total capital at time t is $K(t)$
- Total population (labor force) $L(t)$, which grows at exogenous rate n, e.g. $\dot{L} = nL$
- Homogeneous quantity called GDP denoted $Y(t)$
- GDP can either be consumed $C(t)$ or invested to get $\dot{K}(t)$, or used to replace depreciated capital $\mu K(t)$.

$$Y(t) = C(t) + \dot{K}(t) + \mu K(t)$$

Example: optimal economic growth

- GDP $Y(t)$ is a function of labor $L(t)$, and capital $K(t)$
- The production function $Y(t) = f_2(K, L)$ is homogeneous of degree one, e.g.
$$Y(t) = L(t) f_2(K/L, 1) = L(t) f(K/L)$$
- Hence we normalize all quantities by population L

$$\begin{aligned} y &= Y/L \quad \text{GDP per capita} \\ k &= K/L \quad \text{Capital investment per capita} \\ c &= C/L \quad \text{Consumption per capita} \end{aligned}$$

and write $y(t) = f(k)$ where f is assumed to be a strictly concave, monotonically increasing function, with slope decreasing from ∞ at 0, to 0 at ∞.

Example: optimal economic growth

Consider the rate of per capita investment

$$\dot{k} = \frac{d}{dt}\left(\frac{K}{L}\right) = \frac{\dot{K}}{L} - \left(\frac{K\dot{L}}{L^2}\right) = \frac{\dot{K}}{L} - n\frac{K}{L} = \frac{\dot{K}}{L} - nk$$

using the fact that $\dot{L}/L = n$. Now we assumed that GDP could be expended in one of three ways, leading to

$$Y = C + \dot{K} + \mu K$$

which we also divide by L to obtain

$$y = c + \dot{k} + (\mu + n)k$$

which, when we substitute $y = f(k)$ gives

$$c(t) = f(k) - \dot{k} - (\mu + n)k(t)$$

Example: optimal economic growth

- We want to maximize the total **utility**
- Utility of per capita consumption is $U(c)$. This would also be a strictly concave, monotonically increasing function (according to the law of diminishing marginal utility, i.e. $U''(c) < 0 < U'(c)$).
- Utility in the future is discounted by rate r, e.g. is given by $U(c)e^{-rt}$
- Our control is how much we consume (and hence what is left to invest $\dot{k}$), and the state is the per capita investment $k(t)$.

Example: optimal economic growth

We want to maximize the total **utility** over time, e.g.

$$F\{c\} = \int_0^T U(c)e^{-rt}\,dt$$

subject to

$$c(t) = f(k) - \dot{k} - (\mu+n)k(t)$$

with $k(0) = k_0$, and $k(T) = k_T$.

Substitute c into the functional and we get

$$F\{k\} = \int_0^T U\left(f(k) - \dot{k} - (\mu+n)k(t)\right)e^{-rt}\,dt$$

Example: optimal economic growth

The E-L equations are

$$\frac{d}{dt}\frac{\partial \psi}{\partial \dot{k}} - \frac{\partial \psi}{\partial k} = 0$$

where $\psi(k,\dot{k}) = U\left(f(k) - \dot{k} - (\mu+n)k(t)\right)e^{-rt}$, so

$$\begin{aligned}
-\frac{d}{dt}e^{-rt}\frac{dU}{dc} - e^{-rt}\frac{dU}{dc}\left[\frac{df}{dk} - (\mu+n)\right] &= 0 \\
-e^{-rt}\frac{d}{dt}\frac{dU}{dc} + e^{-rt}\frac{dU}{dc}\left[r - \frac{df}{dk} + (\mu+n)\right] &= 0 \\
-e^{-rt}\frac{d^2U}{dc^2}\frac{dc}{dt} + e^{-rt}\frac{dU}{dc}\left[r - \frac{df}{dk} + (\mu+n)\right] &= 0
\end{aligned}$$

Example: optimal economic growth

We know $e^{-rt} \neq 0$, so we divide it out, and rearrange to get

$$\frac{dc}{dt} = \left[r + \mu + n - \frac{df}{dk}\right]\frac{U'}{U''}$$

which together with

$$\dot{k} = f(k) - c(t) - (\mu+n)k(t)$$

determines the optimal solution of the system. Remember we are given

- U the utility
- f the per capita production as a function of capital

Example: optimal economic growth

Example, $U(c) = \log(c)$, then $U' = 1/c$ and $U'' = -1/c^2$, so

$$\frac{dc}{dt} = \alpha c \text{ where } \alpha = -\left[r + \mu + n - \frac{df}{dk}\right]$$

so

$$c(t) = Ae^{\alpha t}$$

To solve for k, take linear production model, e.g. $y = \beta k$, and then

$$\dot{k} = \gamma k(t) - c(t) \text{ where } \gamma = (\beta - \mu - n)$$

So

$$k(t) = Be^{\gamma t} + \frac{c(t)}{\gamma - \alpha} = Be^{\gamma t} + \frac{c(t)}{r}$$

with A and B determined by $k(0) = k_0$, and $k(T) = k_T$.

Example: optimal economic growth

To maintain constant consumption $c(t)$ we require $\dot{c} = 0$, and so we must have

$$\frac{df}{dk} = r + \mu + n$$

To maintain constant investment, we require

$$\dot{k} = f(k) - c(t) - (\mu + n)k(t) = 0$$

which together determine a solution (c^*, k^*), where the system is in equilibrium.

For the example $y = \beta k$

$$k = \frac{r + \mu + n}{\beta} \quad \text{and} \quad c = (\beta - \mu - n)k$$