Variational Methods & Optimal Control

lecture 15

Matthew Roughan
<matthew.roughan@adelaide.edu.au>

Discipline of Applied Mathematics
School of Mathematical Sciences
University of Adelaide

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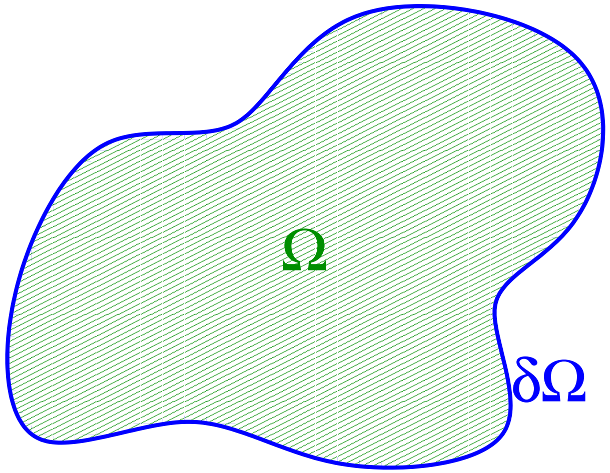
Isoperimetric constraints (continued)

We solve the more general case of Dido's problem: a general shape, without a coast, so that the perimeter must be parametrically described.

Isoperimetric problems



Dido's problem - traditional

Dido's problem is usually posed as follow

Find the curve of length L which encloses the largest possible area, i.e. maximize

$$\text{Area} = \iint_{\Omega} 1 \, dx \, dy$$

subject to the constraint

$$\oint_{\delta\Omega} 1 \, ds = L$$

Of course the problem is not yet in a convenient form.

Green's theorem

Green's theorem converts an integral over the area Ω to a contour integral around the boundary $\delta\Omega$.

$$\iint_\Omega \left(\frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial y} \right) dx\, dy = \oint_{\delta\Omega} \phi\, dy - \psi\, dx$$

for $\phi, \psi : \bar{\Omega} \to I\!R$ such that ϕ, ψ, ϕ_x and ψ_y are continuous.

This converts an area integral over a region into a line integral around the boundary.

Geometric representation of area

The area of a region is given by

$$\text{Area} = \iint_\Omega 1\, dx\, dy$$

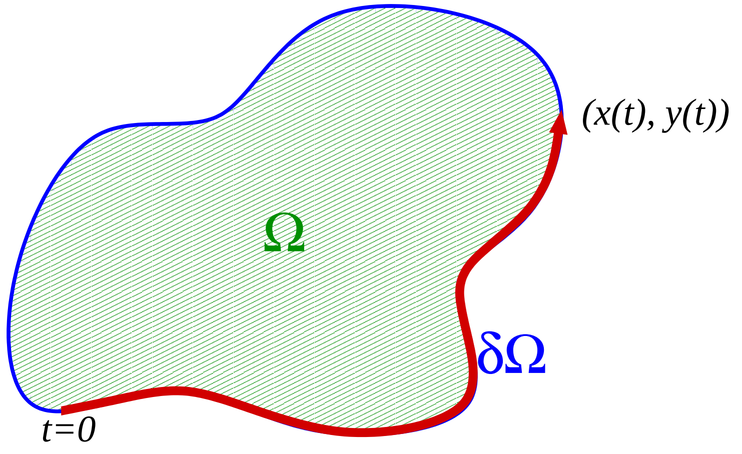
In Green's theorem choose $\phi = x/2$ and $\psi = y/2$, so that we get

$$\text{Area} = \iint_\Omega 1\, dx\, dy = \frac{1}{2} \oint_{\delta\Omega} x\, dy - y\, dx$$

Previous approach to Dido, was to use $y = y(x)$, but in more general case where the boundary must be closed, we can't define y as a function of x (or visa versa). So we write the boundary curve parametrically as $(x(t), y(t))$.

Parametric description of boundary

Boundary $\delta\Omega$ represented parametrically by $(x(t), y(t))$



Dido's problem

If the boundary $\delta\Omega$ is represented parametrically by $(x(t), y(t))$ then

$$\begin{aligned} \text{Area} &= \iint_\Omega 1\, dx\, dy \\ &= \frac{1}{2} \oint_{\delta\Omega} x\, dy - y\, dx \\ &= \frac{1}{2} \oint_{\delta\Omega} x\dot{y} - y\dot{x}\, dt \end{aligned}$$

So now the problem is written in terms of

$$\begin{aligned} \text{one independent variable} &= t \\ \text{two dependent variables} &= (x, y) \end{aligned}$$

Isoperimetric constraint

Previously we wrote the isoperimetric constraint as

$$G\{y\} = \int 1 ds = \int_{x_0}^{x_1} \sqrt{1+y'^2}\, dx = L$$

but now we must also modify this using

$$\frac{ds}{dt} = \sqrt{\frac{dx}{dt}^2 + \frac{dy}{dt}^2}$$

to get

$$G\{x,y\} = \oint 1 ds = \oint \sqrt{\dot{x}^2 + \dot{y}^2} dt = L$$

Dido's problem: Lagrange multiplier

Hence, we look for extremals of

$$H\{x,y\} = \oint \frac{1}{2}\left(x\dot{y} - y\dot{x}\right) + \lambda\sqrt{\dot{x}^2 + \dot{y}^2}\, dt$$

So $h(t,x,y,\dot{x},\dot{y}) = \frac{1}{2}\left(x\dot{y} - y\dot{x}\right) + \lambda\sqrt{\dot{x}^2 + \dot{y}^2}$, and there are two dependent variables, with derivatives

$$\begin{aligned} \frac{\partial h}{\partial x} &= \tfrac{1}{2}\dot{y} & \qquad \frac{\partial h}{\partial \dot{x}} &= -\frac{1}{2}y + \frac{\lambda \dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}} \\ \frac{\partial h}{\partial y} &= -\tfrac{1}{2}\dot{x} & \qquad \frac{\partial h}{\partial \dot{y}} &= \frac{1}{2}x + \frac{\lambda \dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}} \end{aligned}$$

Dido's problem: EL equations

Leading to the 2 Euler-Lagrange equations

$$\begin{aligned} \frac{d}{dt}\left[-\frac{1}{2}y + \frac{\lambda \dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}}\right] &= \frac{1}{2}\dot{y} \\ \frac{d}{dt}\left[\frac{1}{2}x + \frac{\lambda \dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}}\right] &= -\frac{1}{2}\dot{x} \end{aligned}$$

Integrate

$$\begin{aligned} -\frac{1}{2}y + \frac{\lambda \dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}} &= \frac{1}{2}y + A \\ \frac{1}{2}x + \frac{\lambda \dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}} &= -\frac{1}{2}x - B \end{aligned}$$

Dido's problem: solution

$$\begin{aligned} \frac{\lambda \dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}} &= y + A \\ \frac{\lambda \dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}} &= -x - B \end{aligned}$$

Now square the two, and add them to get

$$\lambda^2 \frac{\dot{x}^2 + \dot{y}^2}{\dot{x}^2 + \dot{y}^2} = (y+A)^2 + (x+B)^2$$

or, more simply $(y+A)^2 + (x+B)^2 = \lambda^2$, the equation of a circle with center $(-A,-B)$, radius $|\lambda|$

End-conditions

Note, we can't set value at end points arbitrarily.

- if $x(t_0) = x(t_1)$, and $y(t_0) = y(t_1)$, then we get a closed curve, obviously a circle.
  - these conditions only amount to setting one constant, λ
  - there are many valid circles through (x_0, y_0), with centered along a circle of radius $|\lambda|$ about (x_0, y_0).
- on the other hand, if we specify different end-points, we are really solving a problem such as the simplified problem considered last week.

Why does it work?

Why does the Lagrange multiplier approach work here?

Consider Euler's finite difference method on a uniform grid for approximation of the functional

$$F\{y\} = \int_a^b f(x, y, y')\,dx \simeq \sum_{i=1}^{n} f\left(x_i, y_i, \frac{\Delta y_i}{\Delta x}\right)\Delta x = \bar{F}(\mathbf{y})$$

where $\Delta x = (b-a)/n$, and $\Delta y_i = y_i - y_{i-1}$. The problem of finding an extremal curve now becomes one of finding stationary points of the function $\bar{F}(y_1, y_2, \ldots, y_n)$.

- we solve this by looking for $\partial \bar{F}/\partial y_i = 0$ for all $i = 1, 2, \ldots, n$.

Why does it work?

The constraint can be likewise approximated to give

$$G\{y\} \simeq \sum_{i=1}^{n} g\left(x_i, y_i, \frac{\Delta y_i}{\Delta x}\right)\Delta x = \bar{G}(\mathbf{y}) = L$$

Under our usual conditions on F and G, the limit as $n \to \infty$ gives

$$\begin{aligned} \bar{F}(\mathbf{y}) &\rightarrow F\{y\} \\ \bar{G}(\mathbf{y}) &\rightarrow G\{y\} \end{aligned}$$

That is, the **functions** of the approximation $\mathbf{y}$ converge to the **functionals** of the curve $y(x)$.

Why does it work?

In the finite dimensional case the constraint is

$$\bar{G}(y_1, y_2, \ldots, y_n) - L = 0$$

we use a standard Lagrange multiplier

$$\bar{H}(y_1, y_2, \ldots, y_n, \lambda) = \bar{F}(y_1, y_2, \ldots, y_n) + \lambda\left[\bar{G}(y_1, y_2, \ldots, y_n) - L\right]$$

- we solve this by looking for

$$\frac{\partial \bar{H}}{\partial y_i} = 0, \;\; \forall i = 1, 2, \ldots, n, \quad \text{and} \quad \frac{\partial \bar{H}}{\partial \lambda} = 0$$

- last equation just gives you back your constraint

Why does it work?

In our formulation of the isoperimetric problem we take

$$H\{y\} = F\{y\} + \lambda G\{y\}$$

and we also have

$$\bar{H}(\mathbf{y},\lambda) = \bar{F}(\mathbf{y}) + \lambda\left[\bar{G}(\mathbf{y}) - L\right]$$

In the limit as $n \to \infty$ we find that

$$\bar{H}(\mathbf{y},\lambda) \to H\{y\} - \lambda L$$

The E-L equations for $H\{y\} - \lambda L$ and $H\{y\}$ are the same, so they have the same extremals!

Why does it work?

See van Brunt, pp.83–87 for a more rigorous explanation of Lagrange multipliers in this context.

Multiple constraints

We can also handle multiple constraints via multiple Lagrange multipliers. For instance, given we wish to find extremals of

$$F\{y\} = \int_{x_0}^{x_1} f(x,y,y')\,dx$$

with the m constraints

$$G_k\{y\} = \int_{x_0}^{x_1} g_k(x,y,y')\,dx = L_k$$

we would look for extremals of

$$H\{y\} = \int_{x_0}^{x_1} h(x,y,y')\,dx = \int_{x_0}^{x_1} f(x,y,y') + \sum_{k=1}^{m} \lambda_k g_k(x,y,y')\,dx$$