Variational Methods and Optimal Control
Class Exercise 3: due before lecture, on Thursday 6th September, 2012

Matthew Roughan
<matthew.roughan@adelaide.edu.au>

1. **Simple examples:** Find extremals of the following functionals

   (a) $F\{y\} = \int_{-1}^{0} \sqrt{y(1+y'^2)}\, dx$ with end-points $y(-1) = 0$ and $y(0) = 0$.

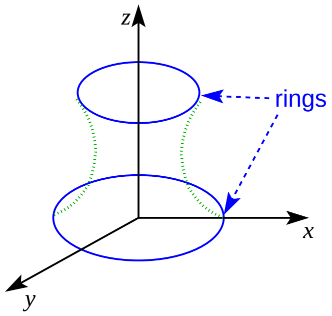
   (b) $F\{y\} = \int_{0}^{1} (y'^2 - y^2 - y)e^{2x}\, dx$ with end-points $y(0) = 0$ and $y(1) = e^{-1}$.

2. **Multiple dependent variables:** Find the form of extremals of the following functionals

   (a) $F\{y(x), z(x)\} = \int_{x_1}^{x_2} (2yz - 2y^2 + y'^2 - z'^2)\, dx$

   (b) $F\{\mathbf{q}(t)\} = \int_{t_1}^{t_2} \dot{q}_1\dot{q}_2 + \dot{q}_2 q_3 + \dot{q}_3 q_1\, dt$

3. **Surface of minimum area:** Consider a soap bubble suspended between two parallel concentric, but displaced rings of radius r_0 and r_1 (see the figure for a clearer view). Ignoring gravity and other external forces, the shape of the soap bubble will minimize the surface area. Use the Calculus of Variations to explain the shape this bubble will take.



   [Hint: rotational symmetry can be used to reduce this to problem with one dependent variable.]

4. **Ritz's Method:** Use Ritz's method to find an approximate, non-trivial solution to the differential equation

$$y'' + \frac{1}{x}y' + \lambda y = 0,$$

   in the domain $x \in [0, 1]$ where $y(0)$ is non-singular and $y(1) = 1$, and hence determine an approximate value of λ that has a solution.

   [Hints: note that the equation can be written in the form

$$\frac{d}{dx}(xy') + \lambda xy = 0,$$

   and find the corresponding integral for which this is the Euler-Lagrange equation.

   Once you have a variational problem, use the trial function

$$y_{\text{trial}} = a + bx^2 + cx^4,$$

   which we have chosen because the solution is expected to be an even function.]

5. **Higher order derivatives:** find the extremal of the following functional

$$J\{y\} = \int_0^1 y''^2 - 240xy\,dx,$$

subject to $y(0) = 0$, $y'(0) = 1/2$, $y(1) = 1$ and $y'(1) = 1/2$.