Transform Methods & Signal Processing

lecture 02

Matthew Roughan
<matthew.roughan@adelaide.edu.au>

Discipline of Applied Mathematics
School of Mathematical Sciences
University of Adelaide

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Continuous Fourier Transforms

Fourier's Theorem is not only one of the most beautiful results of modern analysis, but it is said to furnish an indispensable instrument in the treatment of nearly every recondite question in modern physics.

Lord Kelvin

Jean Baptiste Joseph Fourier

March 21, 1768 — May 16, 1830

- son of a tailor (in Auxerre, France)
  - 12th of 15 children
- involved in the French revolution
  - at one point was arrested
- 1798 Fourier joined Napoleon's army in its invasion of Egypt as scientific adviser
  - helped in archaeological explorations.
- 1802 made Prefect of Grenoble
  - work on heat propagation, and **Fourier series**
- survived Napoleon's arrest, and return, and exile

Fourier series

Can write a periodic function as an (infinite) discrete sum of trigonometric terms, e.g. for period 2π

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$$

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\, dx \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\, dx \end{aligned}$$

Fourier series as a representation

Fourier series is representing the set of functions with period 2π in terms of the basis functions $\cos$ and $\sin$, exploiting orthogonality of these functions

$$\int_{-\pi}^{\pi} \cos(nx)\,dx = 0 \qquad \int_{-\pi}^{\pi} \cos(nx)\cos(mx)\,dx = \pi\delta_{mn}$$

$$\int_{-\pi}^{\pi} \sin(nx)\,dx = 0 \qquad \int_{-\pi}^{\pi} \sin(nx)\sin(mx)\,dx = \pi\delta_{mn}$$

$$\int_{-\pi}^{\pi} \sin(nx)\cos(mx)\,dx = 0$$

δ_{mn} is the Kronecker delta, $\delta_{mn} = \begin{cases} 1, & \text{if } m = n \\ 0, & \text{otherwise.} \end{cases}$

Complex Fourier series

Can write Fourier series in complex form

$$f(x) = \sum_{n=-\infty}^{\infty} A_n e^{inx}$$

$$A_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} \, dx$$

NB: $e^{inx} = \cos(nx) + i \sin(nx)$

Fourier series for other periods

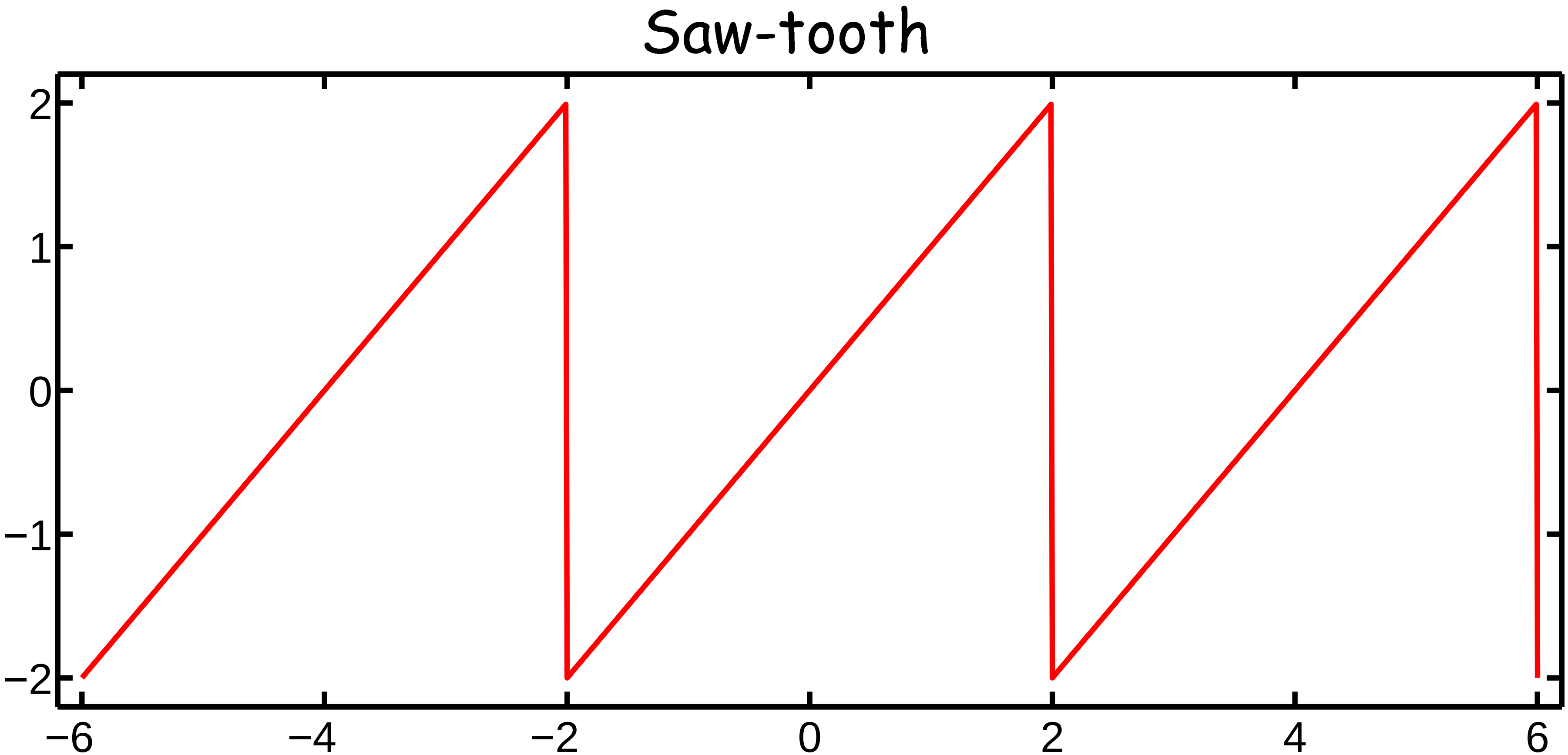
For a function with period L, we need to scale the basis functions by $2\pi/L$

$$f(x) = \sum_{n=-\infty}^{\infty} A_n e^{i2\pi nx/L}$$

$$A_n = \frac{1}{L}\int_{-L/2}^{L/2} f(x) e^{-i2\pi nx/L}\, dx$$

Example Fourier Series

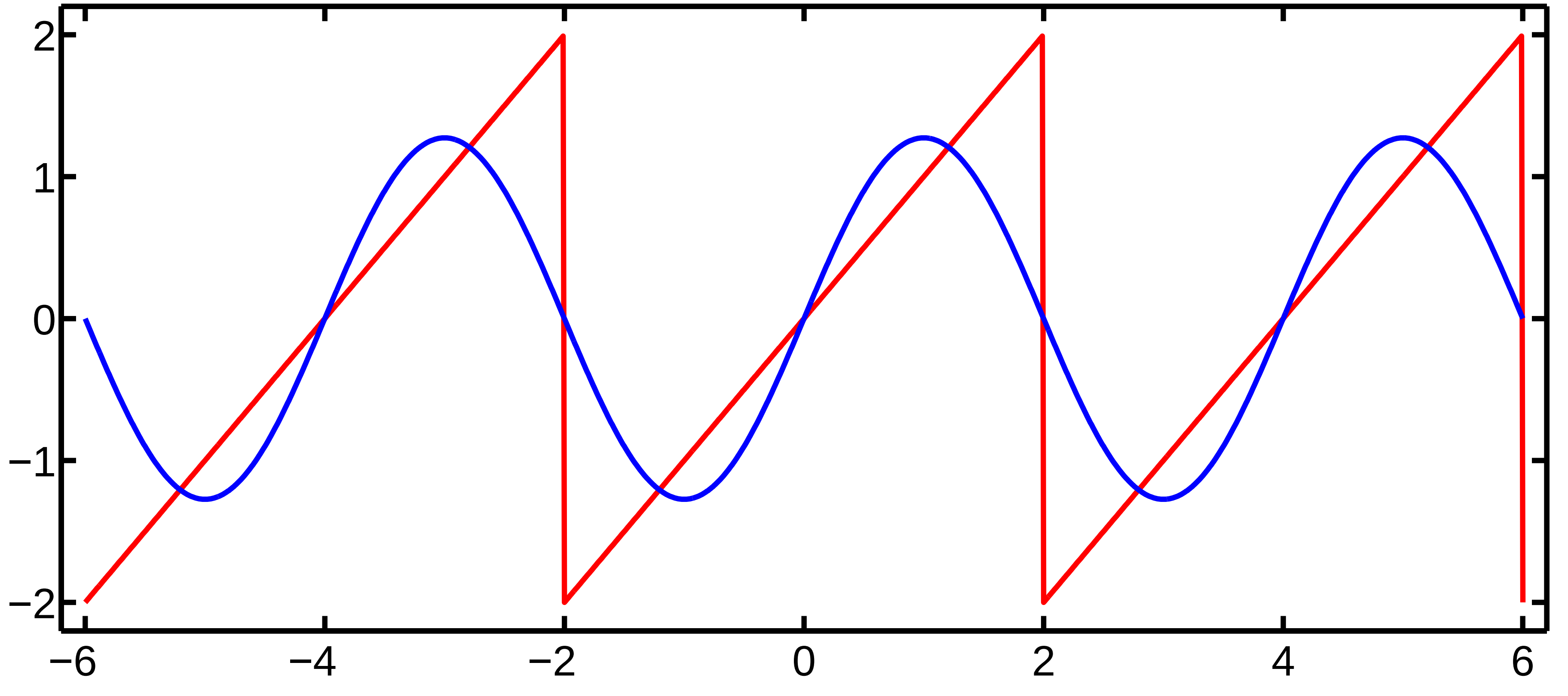
Fourier series for the saw-tooth $f(x) = x$ for $x = [-L, L]$



Example Fourier Series

Fourier series for the saw-tooth $f(x) = x$ for $x = [-L, L]$

1 Fourier component



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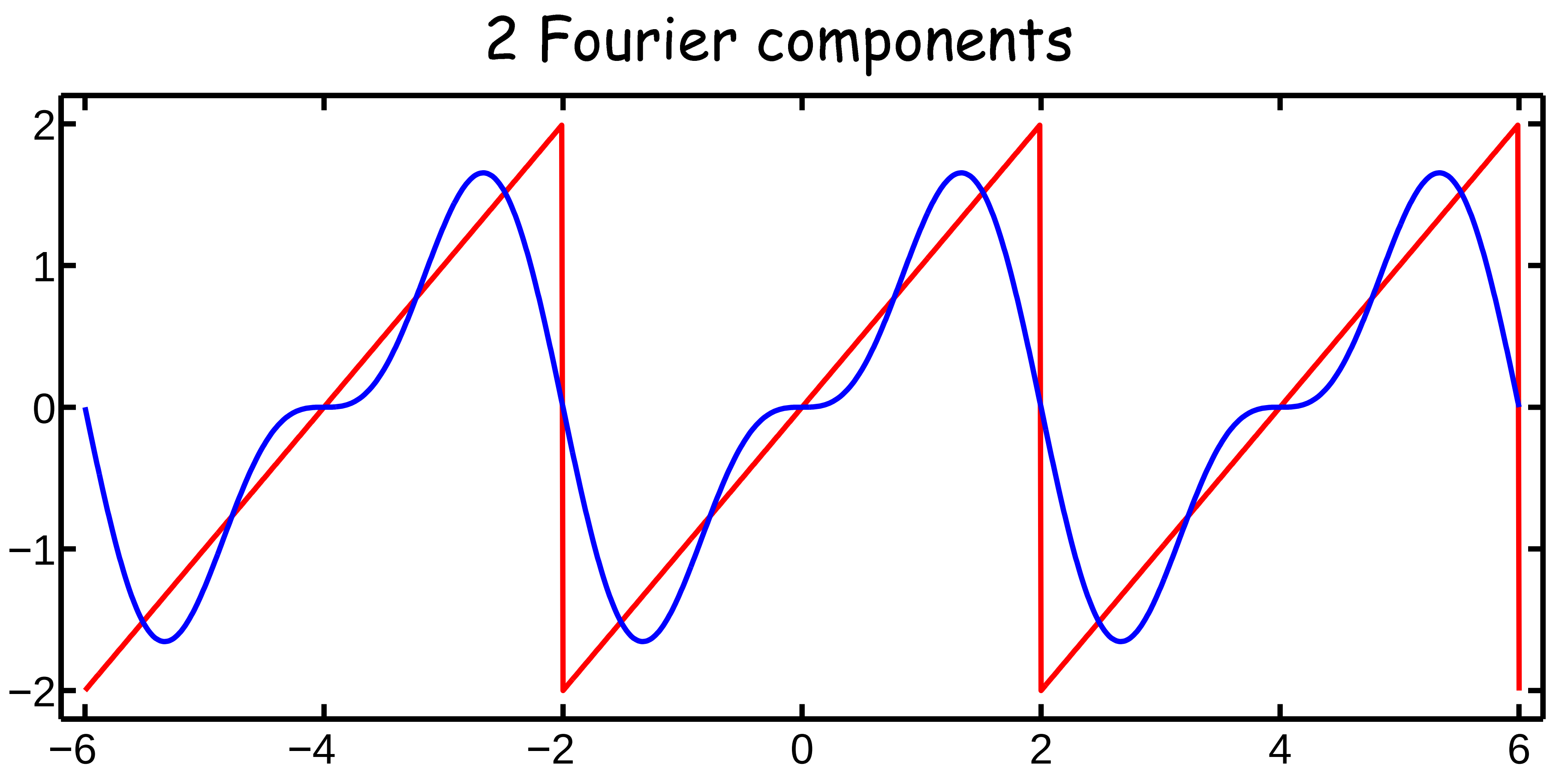
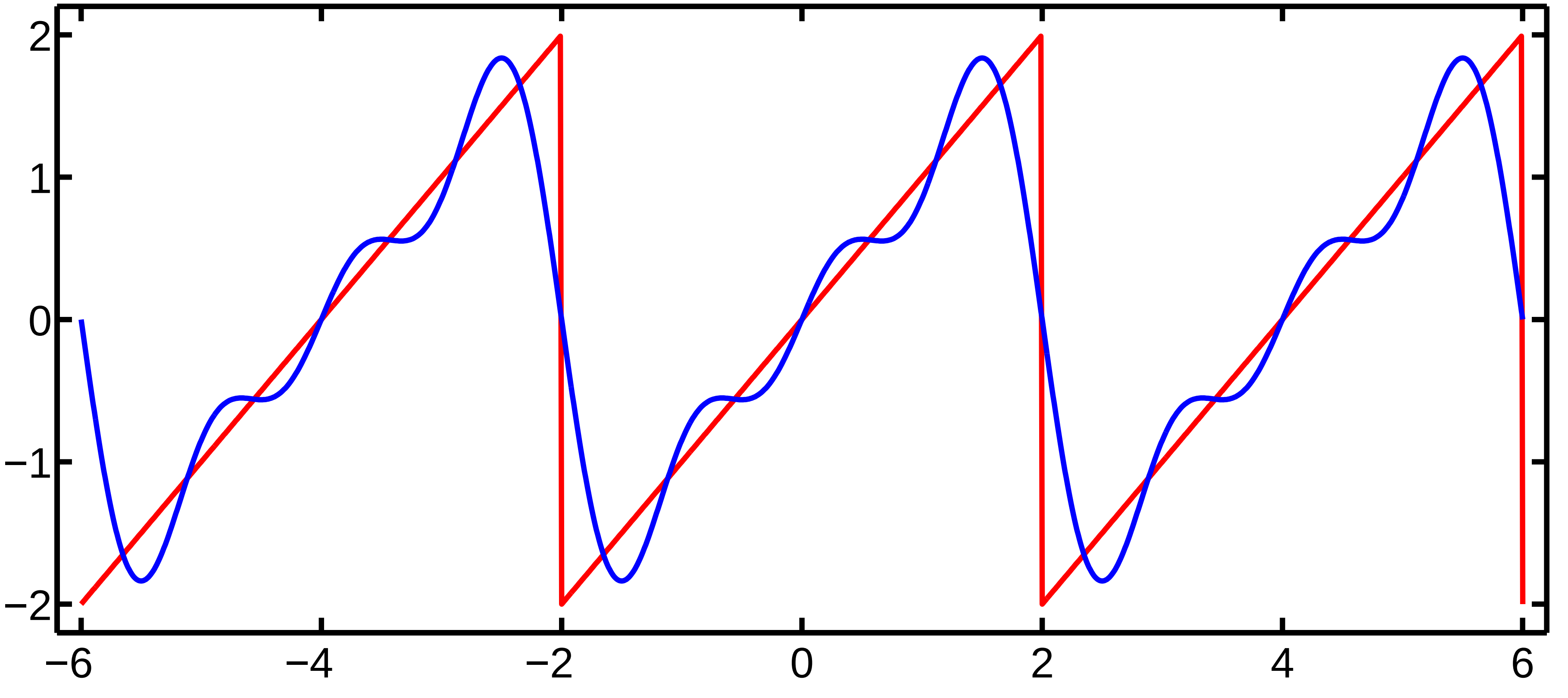


Example Fourier Series

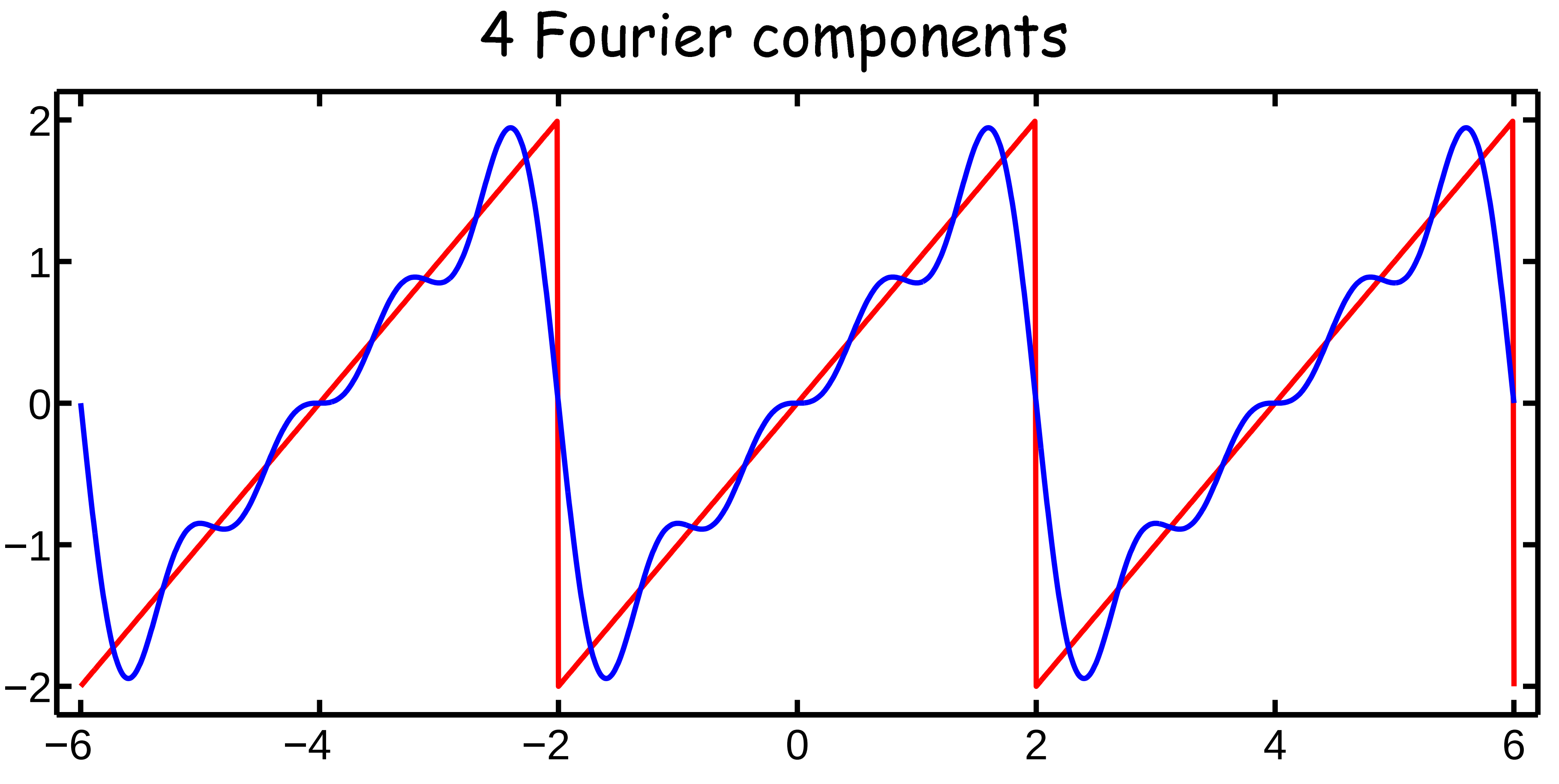
Fourier series for the saw-tooth $f(x) = x$ for $x = [-L, L]$

3 Fourier components



Example Fourier Series

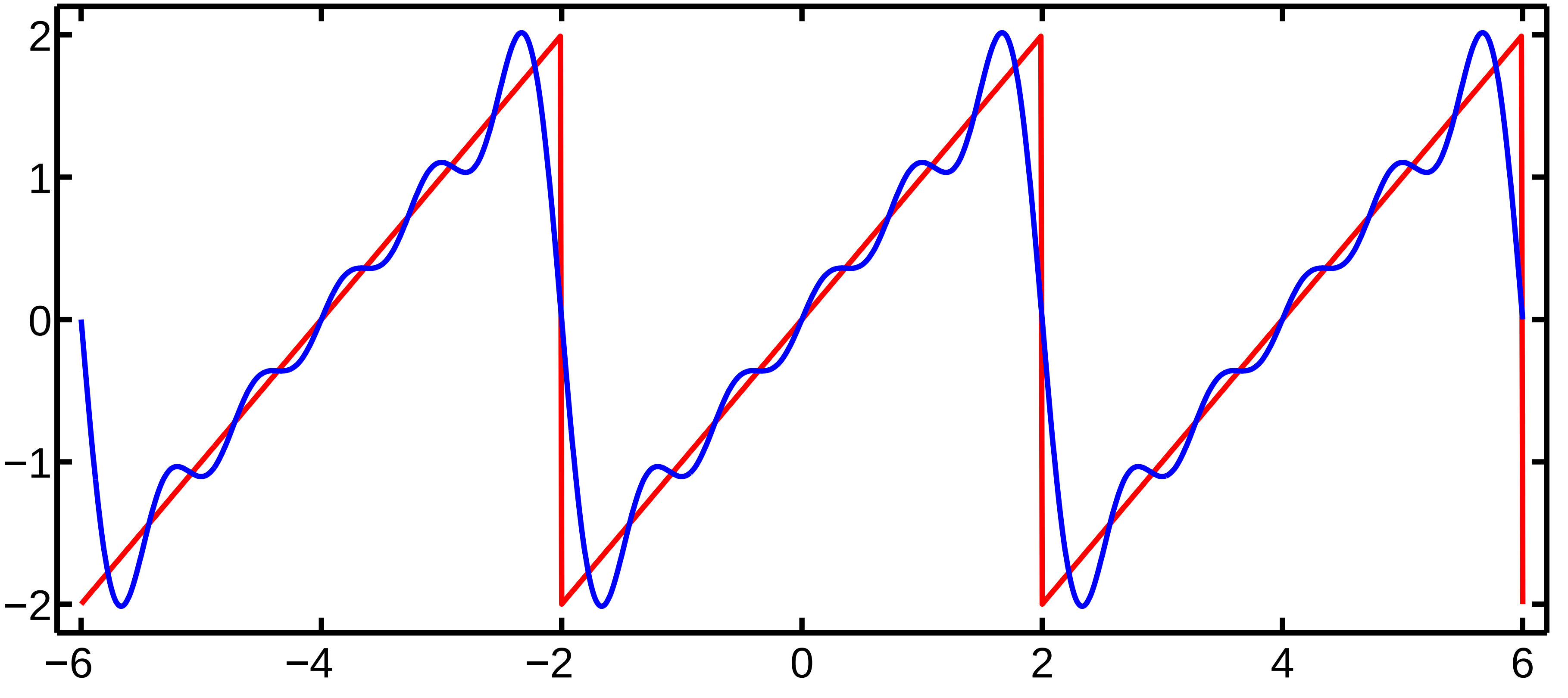
Fourier series for the saw-tooth $f(x) = x$ for $x = [-L, L]$



Example Fourier Series

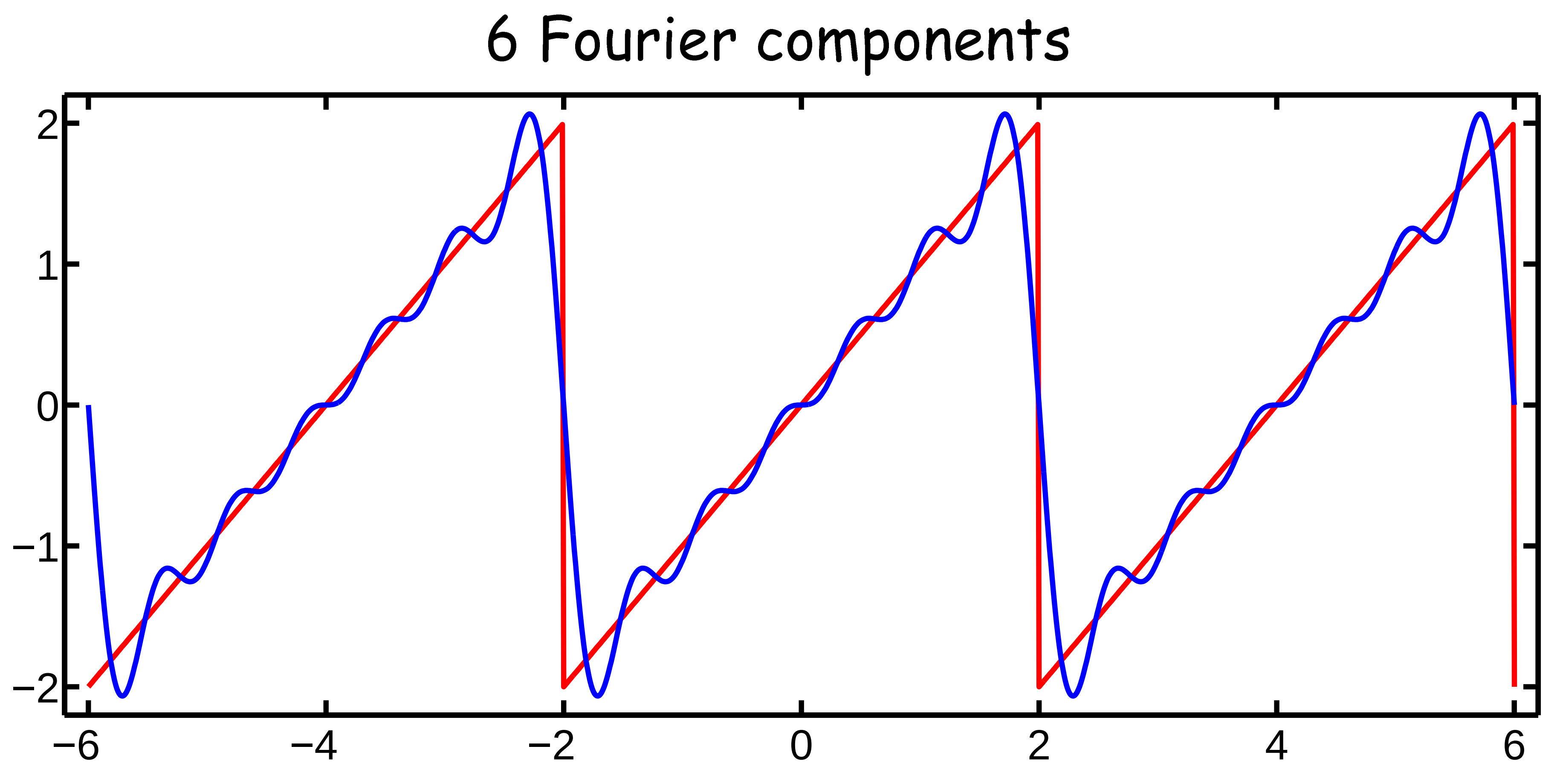
Fourier series for the saw-tooth $f(x) = x$ for $x = [-L, L]$

5 Fourier components



Example Fourier Series

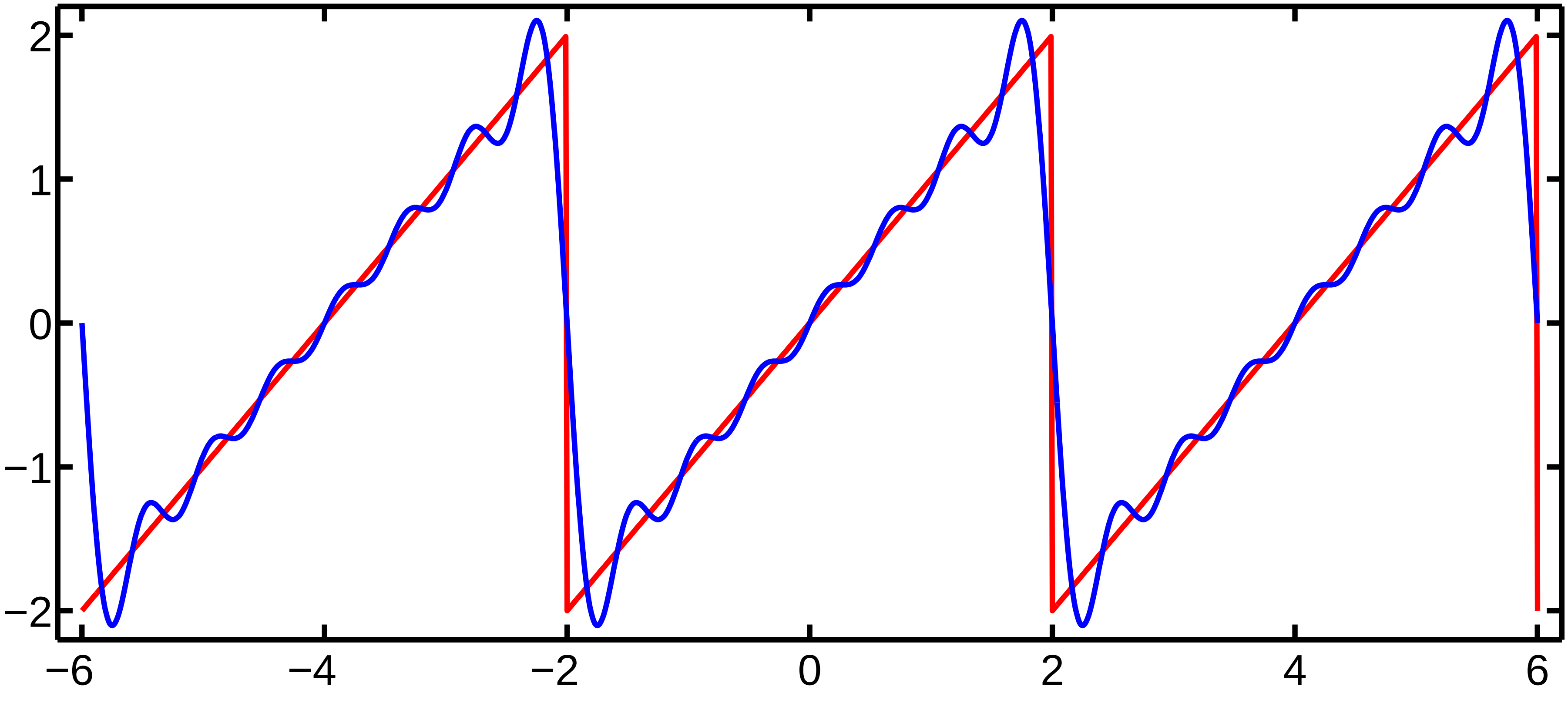
Fourier series for the saw-tooth $f(x) = x$ for $x = [-L, L]$



Example Fourier Series

Fourier series for the saw-tooth $f(x) = x$ for $x = [-L, L]$

7 Fourier components



Integral transforms

- An **integral transform** is a transform defined in terms of an integral

$$f(t) \to \int f(t)g(t,s)dt$$

- Map a function (say of time) to a function of s
- $g(\cdot)$ is called the **kernel** of the transform
- notation (several alternatives)
  - $T\{f(t);s\} = \int f(t)g(t,s)\,dt$
  - $F(s) = \int f(t)g(t,s)\,dt$, $H(s) = \int h(t)g(t,s)\,dt$
  - $\mathcal{F}(s) = \int f(t)g(t,s)\,dt$, $\mathcal{H}(s) = \int h(t)g(t,s)\,dt$
  - $\tilde{f}(s) = \int f(t)g(t,s)\,dt$

Fourier Transform

Fourier transform $F(s) = \int_{-\infty}^{\infty} f(t) e^{-i2\pi st} \, dt$

Inverse transform $f(t) = \int_{-\infty}^{\infty} F(s) e^{i2\pi st} \, ds$

We are writing function $f(t)$ as a continuous integral of trigonometric functions, weighted by $F(s)$.

- think of as a representation of a function
- sines and cosines are forming a basis
- integral transform with kernel function $g(s,t) = e^{-i2\pi st}$

Example: FT of a delta function

From the definition of FT

$$\mathcal{F}\{\delta(t-t_0)\} = \int_{-\infty}^{\infty} \delta(t-t_0)\, e^{-i2\pi st}\, dt$$

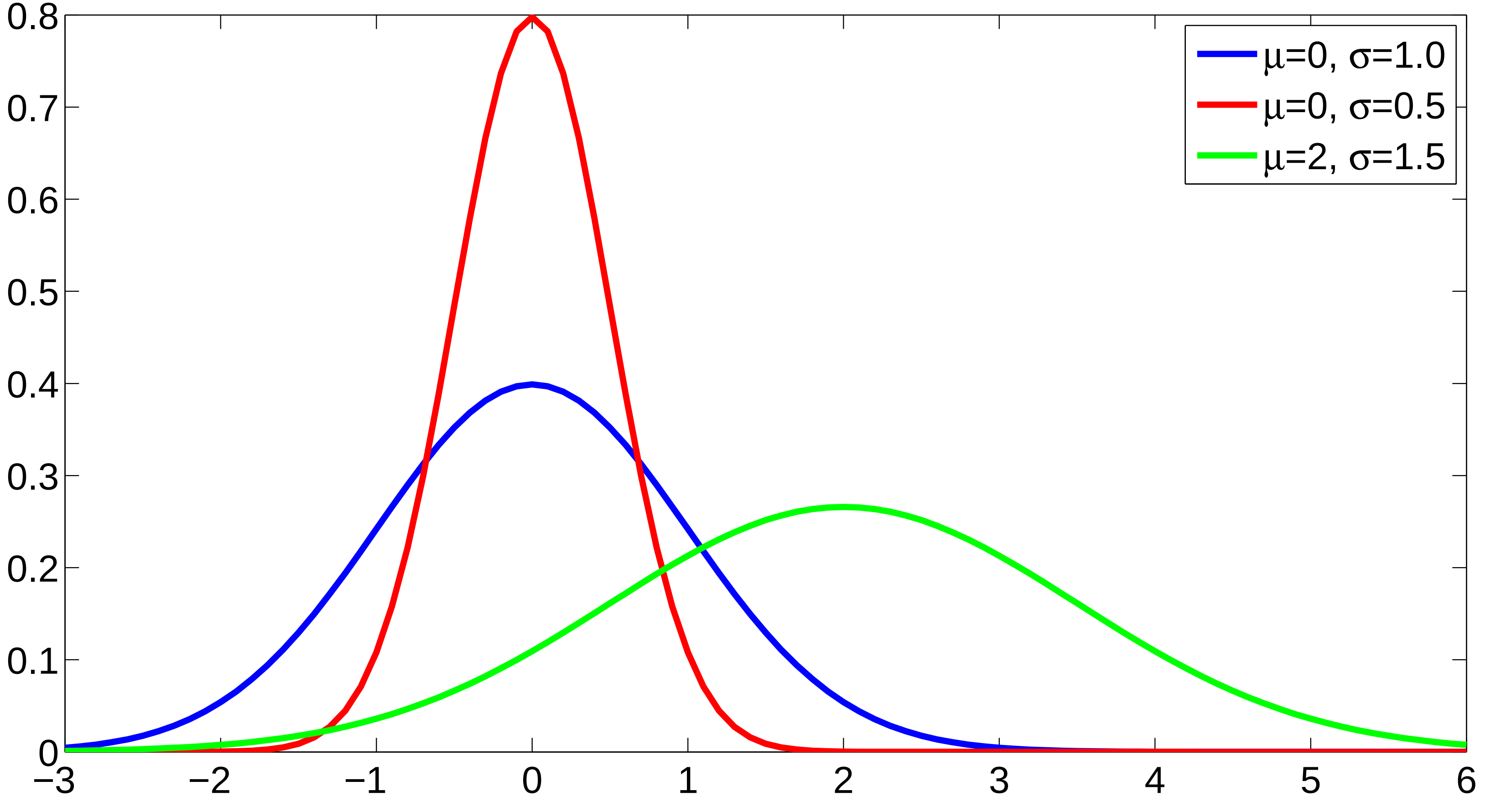
from the definition of a delta

$$\int_{-\infty}^{\infty} f(t)\delta(t-t_0)\, dt = f(t_0)$$

$$= e^{-i2\pi st_0}$$

Gaussian

$$f(x;\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(t-\mu)^2/2\sigma^2}$$



Example: FT of a Gaussian

From the definition of FT

$$
\begin{aligned}
\mathcal{F}\left\{e^{-\pi t^2}\right\} &= \int_{-\infty}^{\infty} e^{-\pi t^2} e^{-i2\pi st}\, dt \\
&= \int_{-\infty}^{\infty} e^{-\pi\left(t^2+i2st\right)}\, dt \\
&= e^{-\pi s^2} \int_{-\infty}^{\infty} e^{-\pi(t+is)^2}\, dt \\
&= e^{-\pi s^2} \int_{-\infty}^{\infty} e^{-\pi u^2}\, du \\
&= e^{-\pi s^2}
\end{aligned}
$$

FT of some simple functions

Function	Transform
$\delta(t)$	1
$\delta(t-t_0)$	$e^{-i2\pi t_0 s}$
$r(t)$	$\mathrm{sinc}(s)$
$e^{-\lvert t\rvert}$	$\frac{2}{4\pi^2 s^2+1}$
$e^{-\pi t^2}$	$e^{-\pi s^2}$

Function	Transform
1	$\delta(t)$
$e^{i2\pi s_0 t}$	$\delta(s-s_0)$
$\mathrm{sinc}(t)$	$r(s)$
$\frac{2}{4\pi^2 t^2+1}$	$e^{-\lvert s\rvert}$
$e^{-\pi t^2}$	$e^{-\pi s^2}$

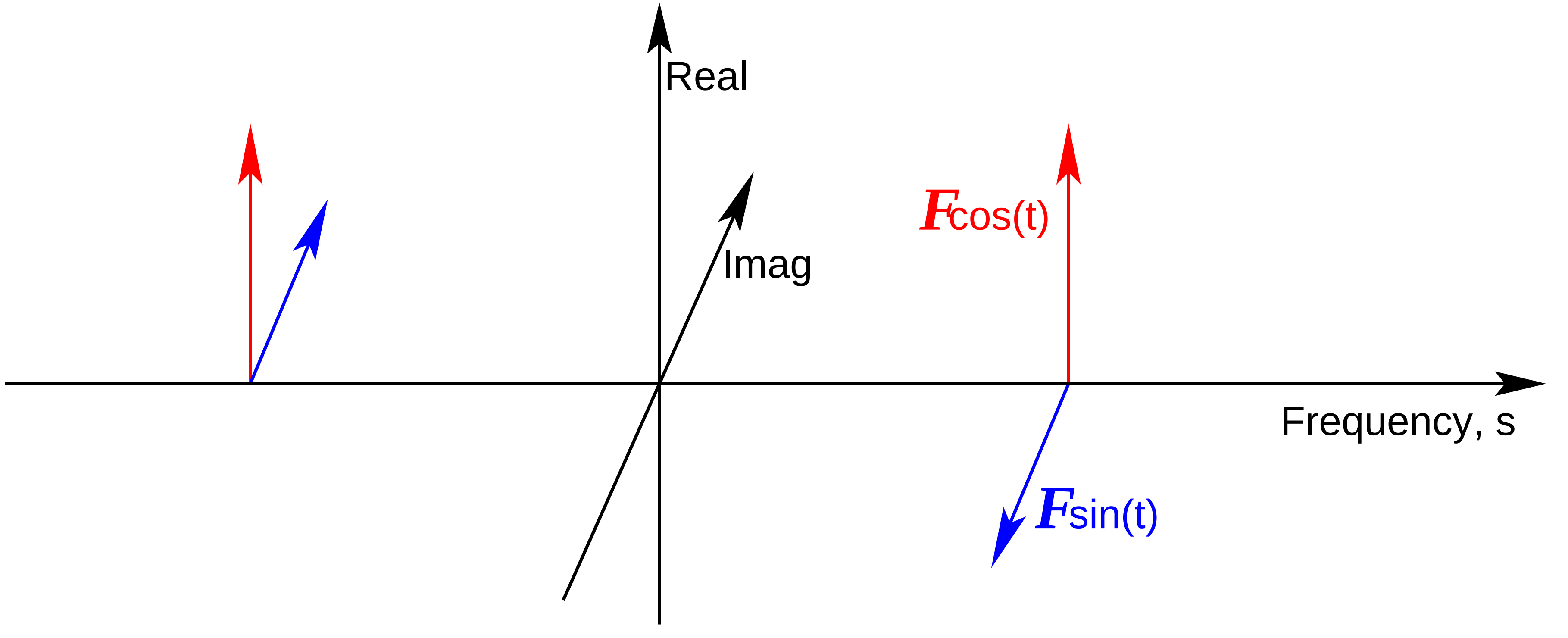
Deriving a Fourier Transform

We can derive a Fourier transform from scratch, but that can sometimes be hard work. It can be easier to use transforms we already know (and their properties).

e.g. exploiting linearity (see later)

$$
\begin{aligned}
\mathcal{F}\left\{\cos(2\pi s_0 t)\right\} &= \int_{-\infty}^{\infty} \cos(2\pi s_0 t)\, e^{-i2\pi st}\, dt \\
&= \mathcal{F}\left\{\frac{1}{2}\left[e^{-i2\pi s_0 x} + e^{i2\pi s_0 x}\right]\right\} \\
&= \frac{1}{2}\mathcal{F}\left\{e^{-i2\pi s_0 x}\right\} + \frac{1}{2}\mathcal{F}\left\{e^{i2\pi s_0 x}\right\} \\
&= \frac{1}{2}\delta(s+s_0) + \frac{1}{2}\delta(s-s_0) \\
\mathcal{F}\left\{\sin(2\pi s_0 t)\right\} &= \frac{i}{2}\delta(s+s_0) - \frac{i}{2}\delta(s-s_0)
\end{aligned}
$$

FTs of sin and cos



The Fourier Transform: definitions

Multiple possible definitions

Fourier transform	**Inverse**
$F(s) = \int_{-\infty}^{\infty} f(t) e^{-i2\pi st} \, dt$	$f(t) = \int_{-\infty}^{\infty} F(s) e^{i2\pi st} \, ds$
$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} \, dt$	$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} \, d\omega$
$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} \, dt$	$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} \, d\omega$

Wave terminology

Definition: Amplitude is the extent of a waves oscillation, e.g. a signal $f(t) = A\sin(t)$ has amplitude $f(t)$.

Definition: Magnitude is the absolute value of amplitude, e.g. for $f(t) = A\sin(t)$ the amplitude is $|f(t)|$.

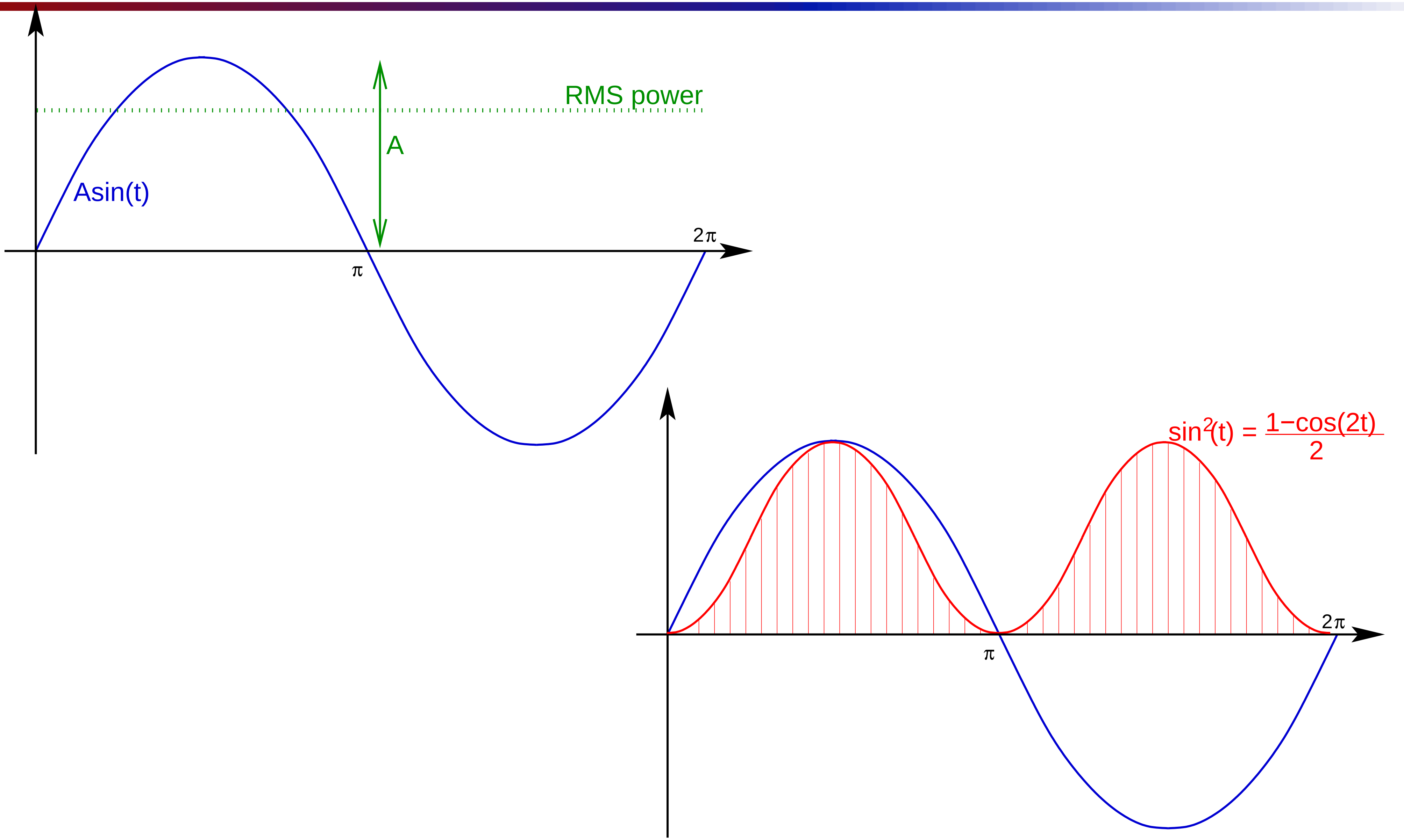
Definition: Power is the square of magnitude, e.g. for $f(t) = A\sin(t)$ the power is $p(t) = |f(t)|^2$.

Definition: RMS Power is the root mean squared power, given by

$$m = \sqrt{\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |f(t)|^2 \, dt}$$

For $f(t) = A\sin(t)$, the RMS power is $A/\sqrt{2}$, e.g. the RMS power of a sin wave is 0.707 times the peak value.

RMS power of a sin wave



RMS power of a sin wave

The sin wave is periodic so we may consider

$$
\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi} |A\sin(t)|^2\, dt &= A^2 \frac{1}{4\pi}\int_{-\pi}^{\pi} 1 - \cos(2t)\, dt \\
&= A^2 \frac{1}{2\pi}\left[\int_{0}^{\pi} 1\, dt - \int_{0}^{\pi} \cos(2t)\, dt\right] \\
&= A^2 \frac{1}{2\pi}\left[\pi - 0\right] \\
&= \frac{A^2}{2}
\end{aligned}
$$

To get the RMS power, take the square root, resulting in $A/\sqrt{2}$.

Measuring power

- power is a square, so can take wide ranging values.
- use a **log scale** to measure
- ear itself 'hears' logarithmically and humans judge the relative loudness of two sounds by the ratio of their intensities, a logarithmic behavior.
- the typical scale used is **Decibels** (deci- from ten, and Bel from Alexander Graham Bell).
- defined WRT a reference power level p_{ref}

$$\text{power} = 10 \log_{10} \frac{p}{p_{\mathrm{ref}}} dB$$

- $p = m^2$, so we may write power $= 20 \log_{10} \frac{m}{m_{\mathrm{ref}}} dB$
- 3 dB corresponds to a factor of 2 in power

Decibels and sounds

Example	Sound Pressure Level (dB)	Sound Intensity (watts/m^2)
Snare drums, played hard at 6 inches	150	1000
30m from jet aircraft	140	100
Threshold of pain	130	10
Jack hammer	120	1
Fender guitar amplifier, full volume at 10 inches	110	0.1
Subway	100	0.01
	90	0.001
Typical home stereo listening level	80	0.0001
Kerbside of busy road	70	0.00001
Conversational speech at 1 foot away	60	10^{-6}
Average office noise	50	10^{-7}
Quiet conversation	40	10^{-8}
Quiet office	30	10^{-9}
Quiet living room	20	10^{-10}
Quiet recording studio	10	10^{-11}
Threshold of hearing for healthy youths	0	10^{-12}

Power Spectra

Definition: The Power Spectrum of a signal $f(t)$ is $|F(s)|^2$, where $F(s)$ is the Fourier transform,

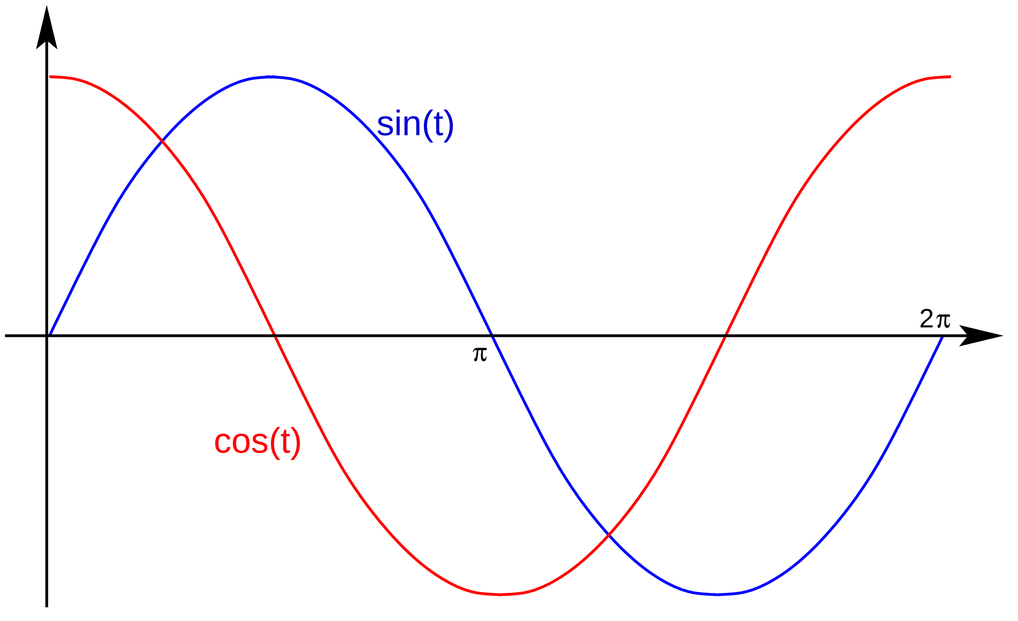
$$F(s) = \int_{-\infty}^{\infty} f(t) e^{-i2\pi st} \, dt$$

- The power spectrum defines the amount of power at each frequency.
- e.g. $|F(0)|^2$ is referred to as the DC term.
- for real-valued signals the power spectrum is even

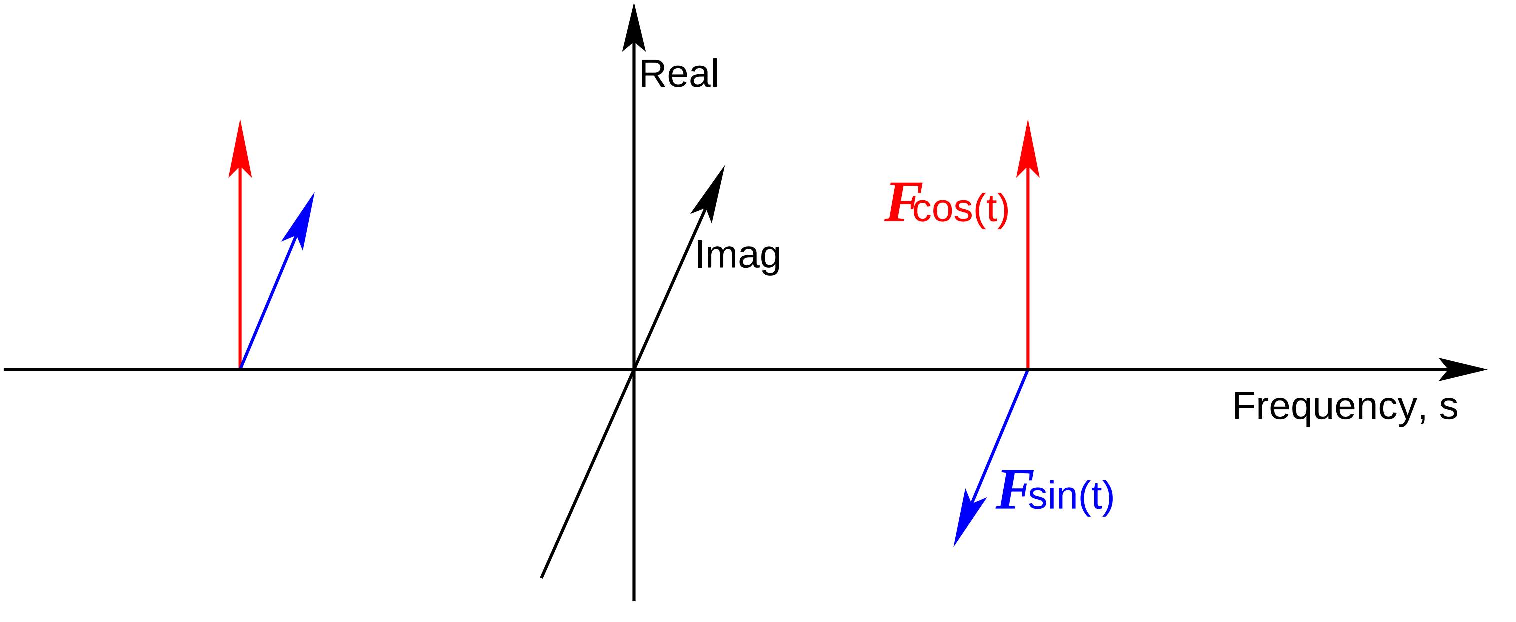
$$|F(-s)|^2 = |F(s)|^2$$

(because the Fourier transform of a real input will be a Hermitian signal).

Phase



- the sin and cosine functions have the same frequency
- $cos(t) = sin(t + \pi/2)$
- there is a phase change of $\pi/2$



Properties of the Fourier transform

Linearity:	$a f_1(t) + b f_2(t)$	$\longrightarrow$	$aF_1(s) + bF_2(s)$
Time shift:	$f(t - t_0)$	$\longrightarrow$	$F(s)e^{-i2\pi s t_0}$
Time scaling:	$f(at)$	$\longrightarrow$	$\frac{1}{\|a\|}F\left(\frac{s}{a}\right)$
Duality:	$F(t)$	$\longrightarrow$	$f(-s)$
Frequency shift:	$f(t)e^{-i2\pi s_0 t}$	$\longrightarrow$	$F(s + s_0)$
Convolution:	$f_1(t) * f_2(t)$	$\longrightarrow$	$F_1(s)F_2(s)$
Differentiation I:	$\frac{d^n}{dt^n} f(t)$	$\longrightarrow$	$(i2\pi s)^n F(s)$
Differentiation II:	$(-i2\pi t)^n f(t)$	$\longrightarrow$	$\frac{d^n}{ds^n} F(s)$
Integration:	$\int_{-\infty}^{t} f(s)ds$	$\longrightarrow$	$\frac{1}{i2\pi s}F(s) + \pi F(0)\delta(s)$

Properties: Linearity

$$\boxed{\mathcal{F}\left\{a f_1(t) + b f_2(t)\right\} = aF_1(s) + bF_2(s)}$$

$$\begin{aligned} \int_{-\infty}^{\infty} [a f_1(t) + b f_2(t)]\, e^{-i2\pi st}\, dt \\ = \; & a \int_{-\infty}^{\infty} f_1(t) e^{-i2\pi st}\, dt + b \int_{-\infty}^{\infty} f_2(t) e^{-i2\pi st}\, dt \\ = \; & aF_1(s) + bF_2(s) \end{aligned}$$

- very useful property
- we can use this to derive Fourier transform, e.g. for cos above
- see more on linearity when we discuss filters (lecture 5-6)

Properties: time shift

$$\mathcal{F}\{f(t-t_0)\} = F(s)e^{-i2\pi s t_0}$$

$$\begin{aligned} \int_{-\infty}^{\infty} f(t-t_0)e^{-i2\pi st}\,dt &= \int_{-\infty}^{\infty} f(t)e^{-i2\pi s(t+t_0)}\,dt \\ &= e^{-i2\pi s t_0}\int_{-\infty}^{\infty} f(t)e^{-i2\pi st}\,dt \\ &= e^{-i2\pi s t_0}F(s) \end{aligned}$$

Note $|F(s)e^{-i2\pi s t_0}| = |F(s)| \times |e^{-i2\pi s t_0}| = |F(s)|$
So the magnitude of the FT is unchanged.

This represents a **phase change**. The higher the frequency, the larger the phase change.

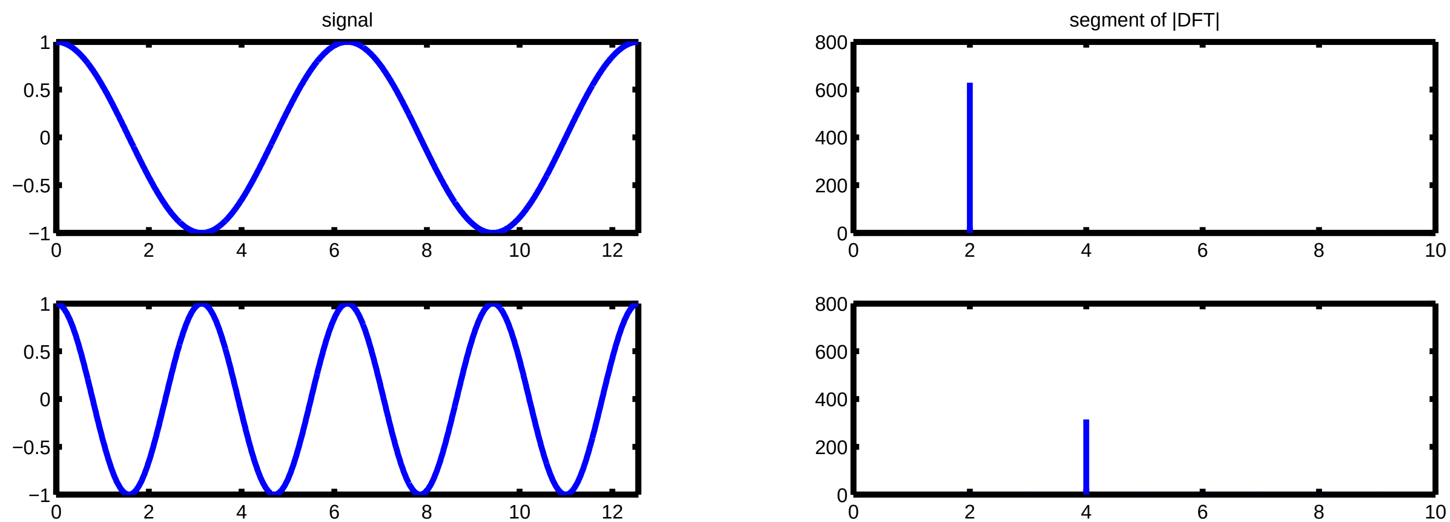
Properties: Time scaling

$$\mathcal{F}\left\{ f(at) \right\} = \frac{1}{|a|} F\left(\frac{s}{a} \right)$$

$$\int_{-\infty}^{\infty} f(at) e^{-i2\pi st}\, dt = \frac{1}{|a|} \int_{-\infty}^{\infty} f(x) e^{-i2\pi(s/a)x}\, dx$$

$$\begin{aligned} x &= at \\ dt &= \frac{1}{a}\, dx \end{aligned}$$

$$= \frac{1}{|a|} F\left(\frac{s}{a} \right)$$



Properties: Duality

$$\boxed{\mathcal{F}\{F(t)\} = f(-s)}$$

Consider the Fourier transform of $F(t)$:

$$\begin{aligned} \int_{-\infty}^{\infty} F(t) e^{-i2\pi st}\, dt &= \int_{-\infty}^{\infty} F(t) e^{i2\pi(-s)t}\, dt, \quad \text{the inverse trans.} \\ &= f(-s) \end{aligned}$$

- the table of Fourier transforms above shows pairs of duals, e.g.

$$\mathcal{F}\{r(t)\} = \mathrm{sinc}(s) \quad \text{and} \quad \mathcal{F}\{\mathrm{sinc}(t)\} = r(t)$$

Properties: Frequency shift

$$\mathcal{F}\left\{ f(t)e^{-i2\pi s_0 t} \right\} = F(s+s_0)$$

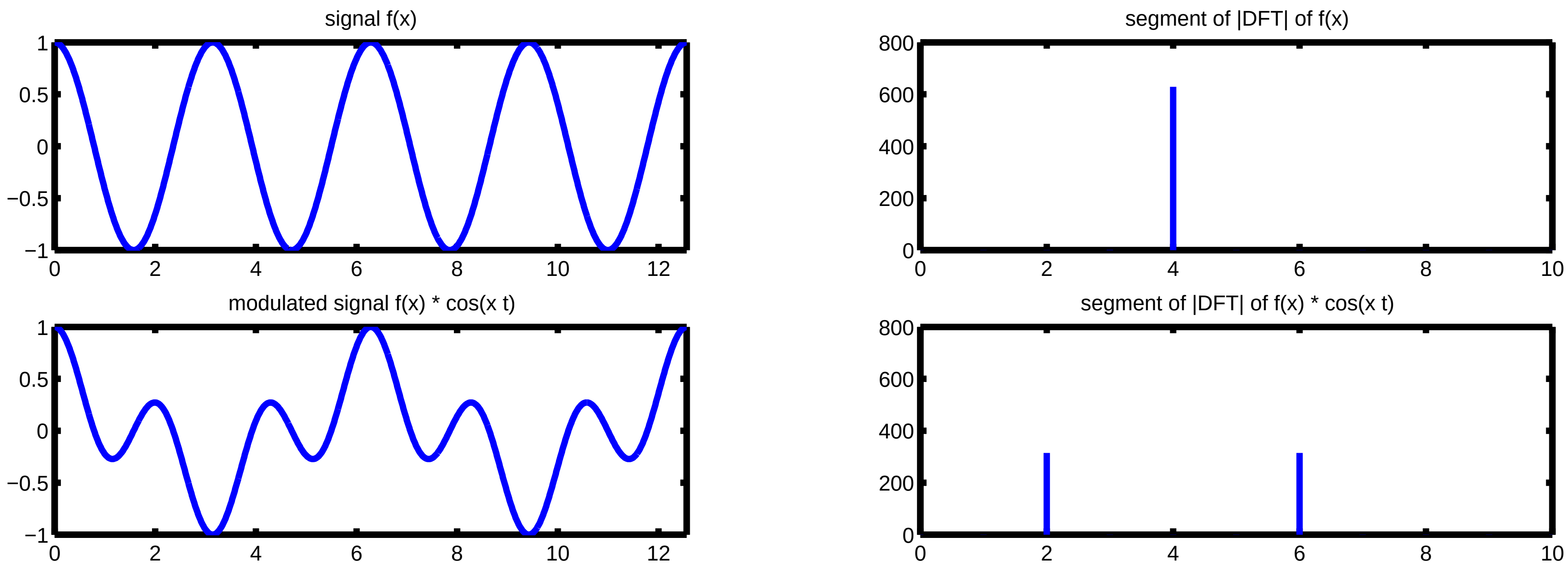
$$\begin{aligned} \int_{-\infty}^{\infty} f(t)e^{-i2\pi s_0 t} e^{-i2\pi s t}\, dt &= \int_{-\infty}^{\infty} f(t)e^{-i2\pi(s+s_0)t}\, dt \\ &= F(s+s_0) \end{aligned}$$

- used for signal modulation, e.g. FM radio
- simpler using a cos function (see below)

Properties: Modulation

$$\mathcal{F}\{f(t)\cos(2\pi s_0 t)\} = \tfrac{1}{2}F(s-s_0) + \tfrac{1}{2}F(s+s_0)$$

For proof, see freq. shift above noting $\cos x = \frac{1}{2}\left(e^{-ix} + e^{ix}\right)$

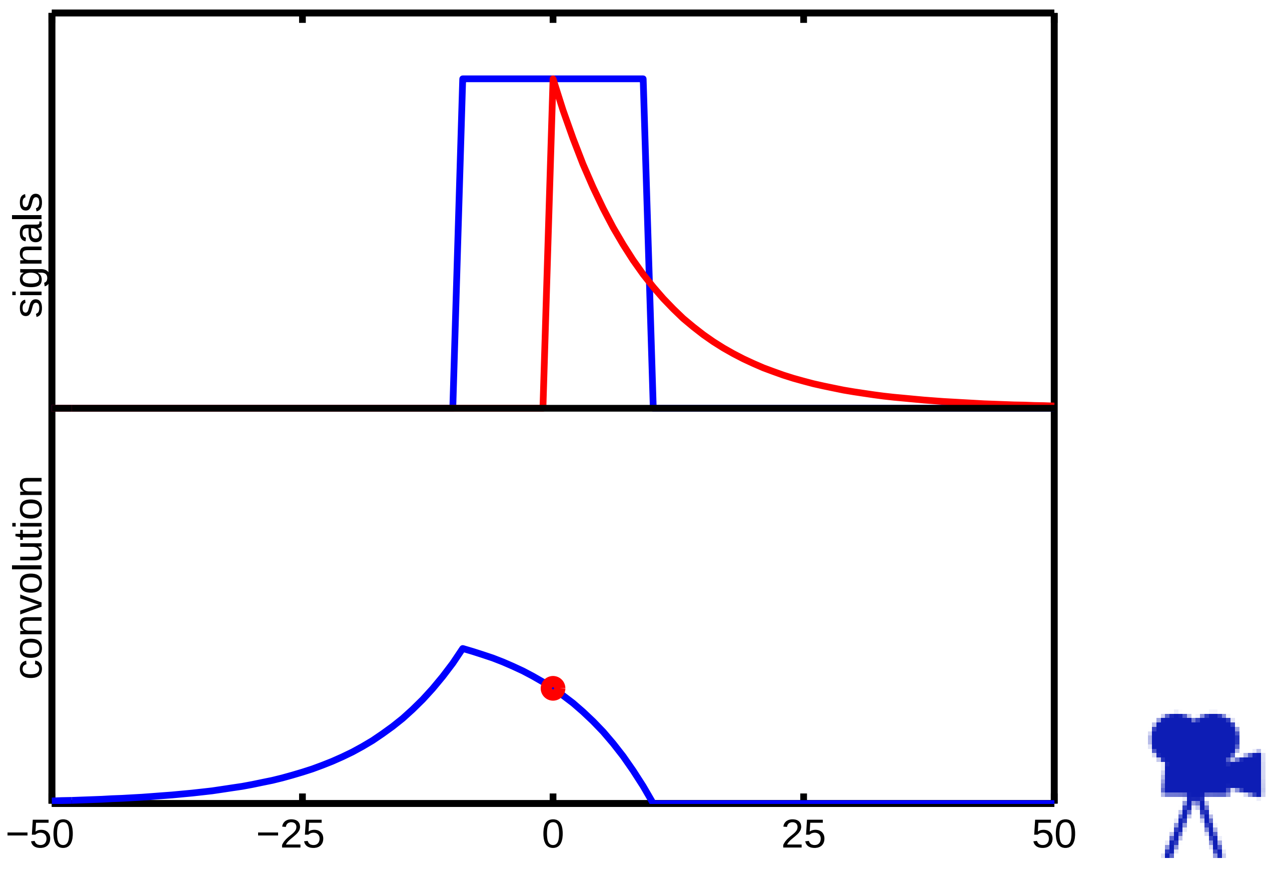


Can use this to generate higher frequency signals, or to demodulate signals.

Convolutions

What is a convolution?

$$f(t) * g(t) = [f * g](t) = \int_{-\infty}^{\infty} f(u)\, g(t-u)\, du$$



Properties: Convolution

$$\boxed{\mathcal{F}\left\{f_1(t) * f_2(t)\right\} \rightarrow F_1(s)F_2(s)}$$

$$\begin{aligned}
\mathcal{F}\left\{f(t) * g(t)\right\} &= \mathcal{F}\left\{\int_{-\infty}^{\infty} f(u)\, g(t-u)\, du\right\} \\
&= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(u)\, g(t-u)\, du\right] e^{-i2\pi st}\, dt \\
&= \int_{-\infty}^{\infty} f(u) \int_{-\infty}^{\infty} g(t-u) e^{-i2\pi st}\, dt\, du \\
&= \int_{-\infty}^{\infty} f(u) G(s) e^{-i2\pi su}\, du \\
&= G(s) \int_{-\infty}^{\infty} f(u) e^{-i2\pi su}\, du \\
&= F(s)G(s)
\end{aligned}$$

Convolution example

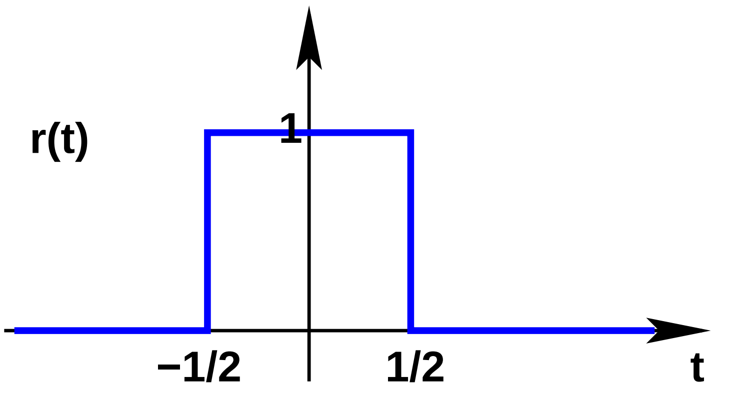
Convolution of two rectangular pulses $r(t)$ where

$r(t) = u(t+1/2) - u(t-1/2)$, and $u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$

$$\begin{aligned} r(t) * r(t) &= \int_{-\infty}^{\infty} r(s)\, r(t-s)\, ds \\ &= \begin{cases} 0, & \text{if } t < -1 \\ \int_{-1/2}^{1/2+t} r(s)\, r(t-s)\, ds, & \text{if } -1 \leq t \leq 0 \\ \int_{t-1/2}^{1/2} r(s)\, r(t-s)\, ds, & \text{if } 0 \leq t \leq 1 \\ 0, & \text{if } t > -1 \end{cases} \end{aligned}$$

Convolution example

For $-1 \leq t \leq 0$, the convolution $r(t) * r(t)$ is



$$
\begin{aligned}
r(t) * r(t) &= \int_{-1/2}^{1/2+t} r(s)\, r(t-s)\, ds, \\
&= \int_{-1/2}^{1/2+t} 1\, ds, \\
&= [t]_{-1/2}^{1/2+t} \\
&= 1/2 + t - -1/2 \\
&= 1 + t,
\end{aligned}
$$

Similarly for $0 \leq t \leq 1$, we get $r(t) * r(t) = 1 - t$

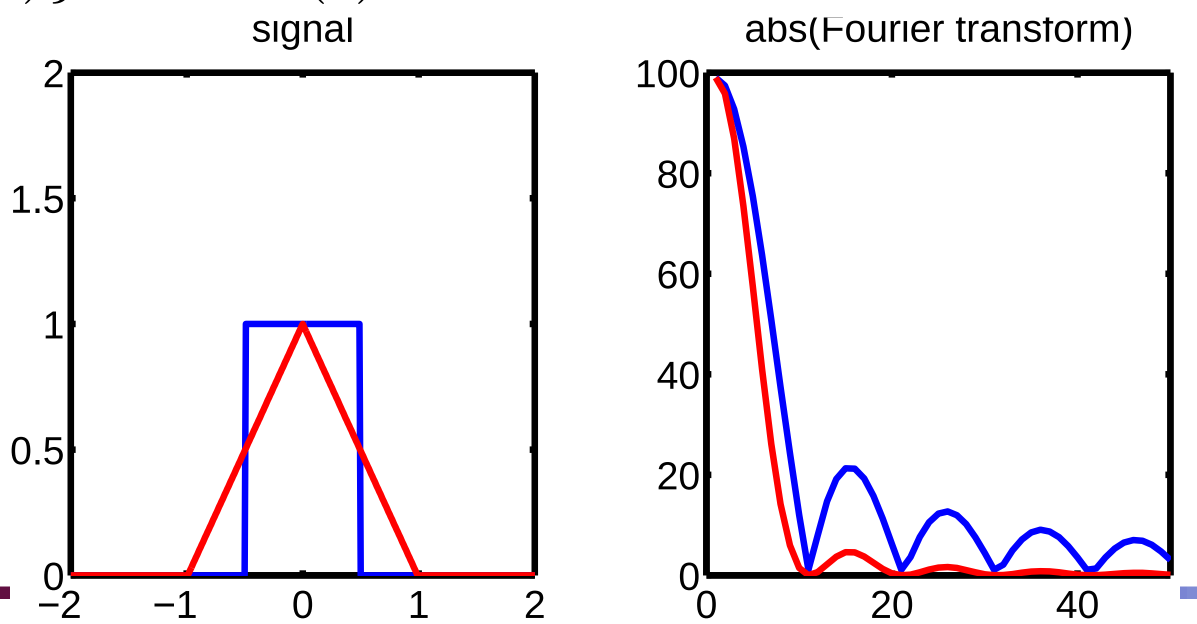
Convolution example

Result is a Triangular pulse

$$
r(t) * r(t) = \begin{cases} 0, & \text{if } t < -1 \\ 1+t, & \text{if } -1 \leq t \leq 0 \\ 1-t, & \text{if } 0 \leq t \leq 1 \\ 0, & \text{if } t > -1 \end{cases}
$$

$\mathcal{F}\{r(t)\} = \text{sinc}(s)$ hence from the convolution theorem

$\mathcal{F}\{r(t) * r(t)\} = \text{sinc}^2(s)$

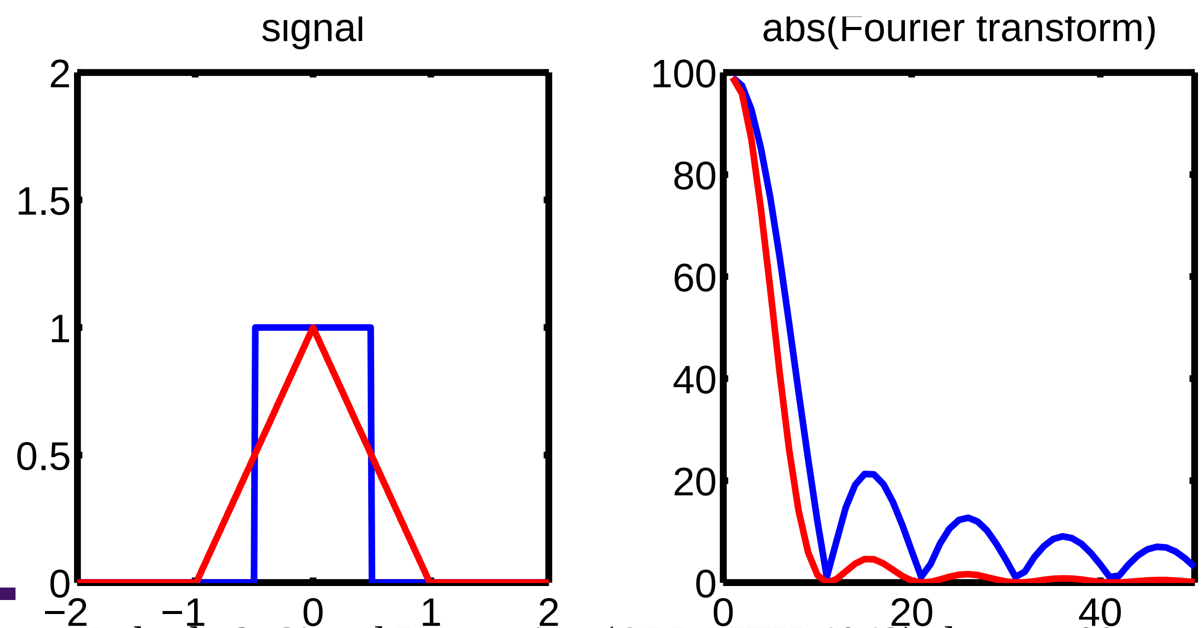


Limiting convolutions

$$\mathcal{F}\{r(t)\} = \text{sinc}(s) \Rightarrow \mathcal{F}\{r(t) * r(t) * \cdots * r(t)\} = \text{sinc}^n(s)$$

- n convolutions of a rectangular pulse produces a function with FT given by $\text{sinc}^n(x)$, which tends to a Gaussian as $n \to \infty$.
- The inverse FT of a Gaussian is also a Gaussian so the limit of $r(t) * r(t) * \cdots * r(t)$ is a Gaussian pulse.

1 convolution

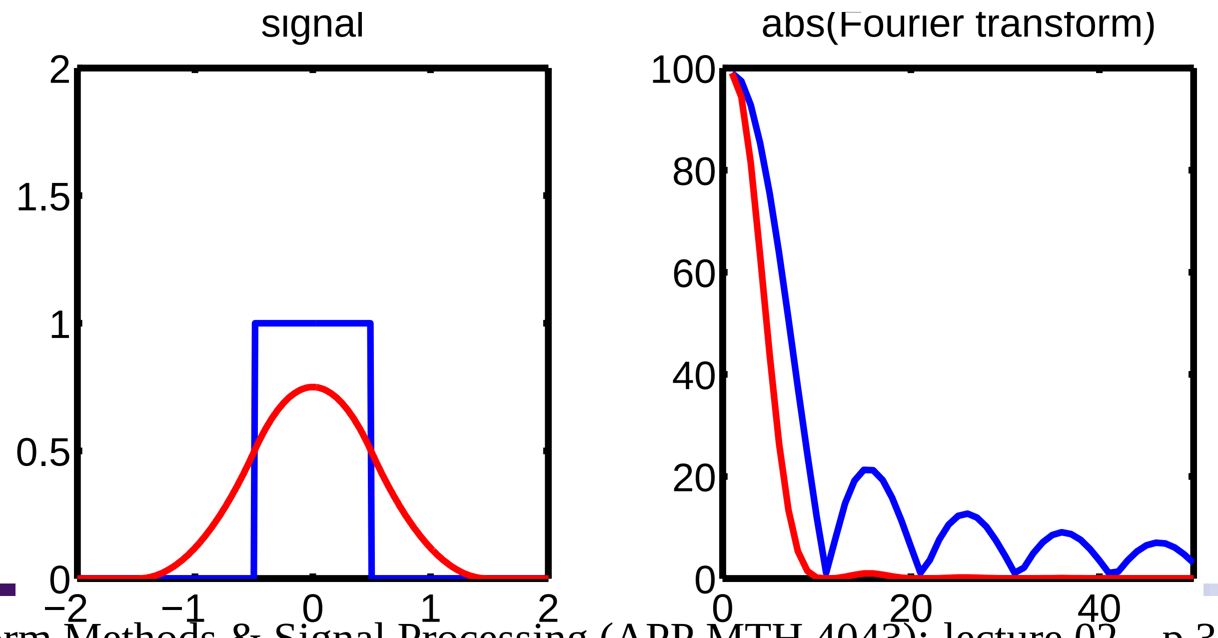


Limiting convolutions

$$\mathcal{F}\{r(t)\} = \text{sinc}(s) \Rightarrow \mathcal{F}\{r(t) * r(t) * \cdots * r(t)\} = \text{sinc}^n(s)$$

- n convolutions of a rectangular pulse produces a function with FT given by $\text{sinc}^n(x)$, which tends to a Gaussian as $n \to \infty$.
- The inverse FT of a Gaussian is also a Gaussian so the limit of $r(t) * r(t) * \cdots * r(t)$ is a Gaussian pulse.

2 convolutions

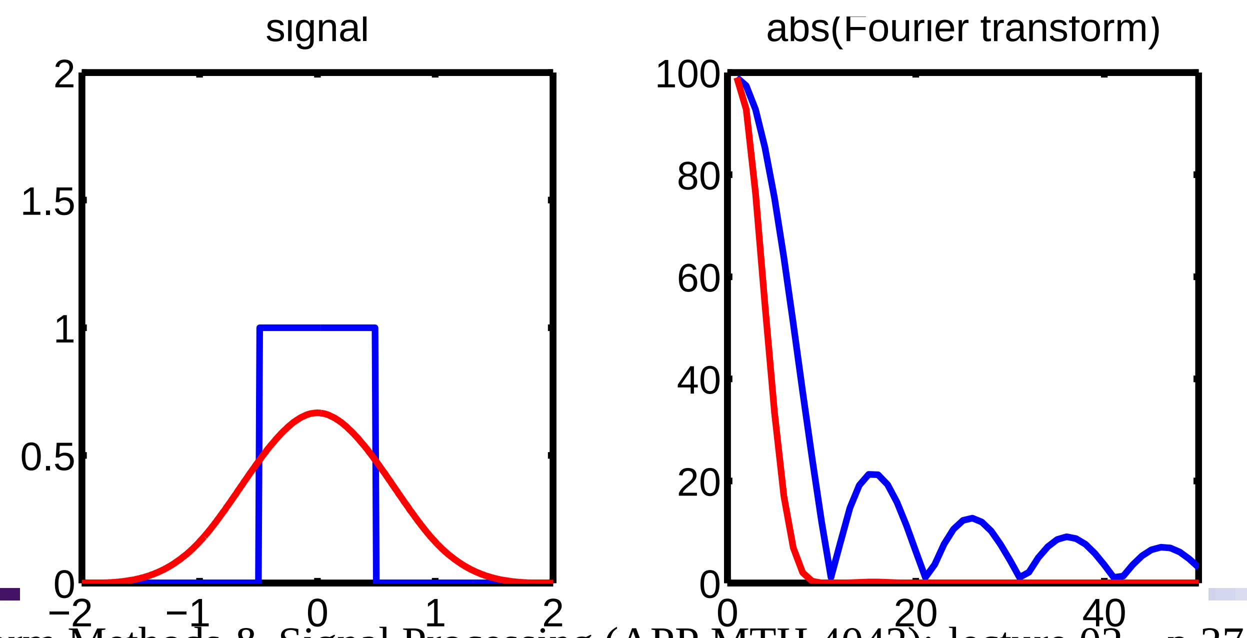


Limiting convolutions

$$\mathcal{F}\{r(t)\} = \operatorname{sinc}(s) \Rightarrow \mathcal{F}\{r(t) * r(t) * \cdots * r(t)\} = \operatorname{sinc}^n(s)$$

- n convolutions of a rectangular pulse produces a function with FT given by $\operatorname{sinc}^n(x)$, which tends to a Gaussian as $n \to \infty$.
- The inverse FT of a Gaussian is also a Gaussian so the limit of $r(t) * r(t) * \cdots * r(t)$ is a Gaussian pulse.

3 convolutions

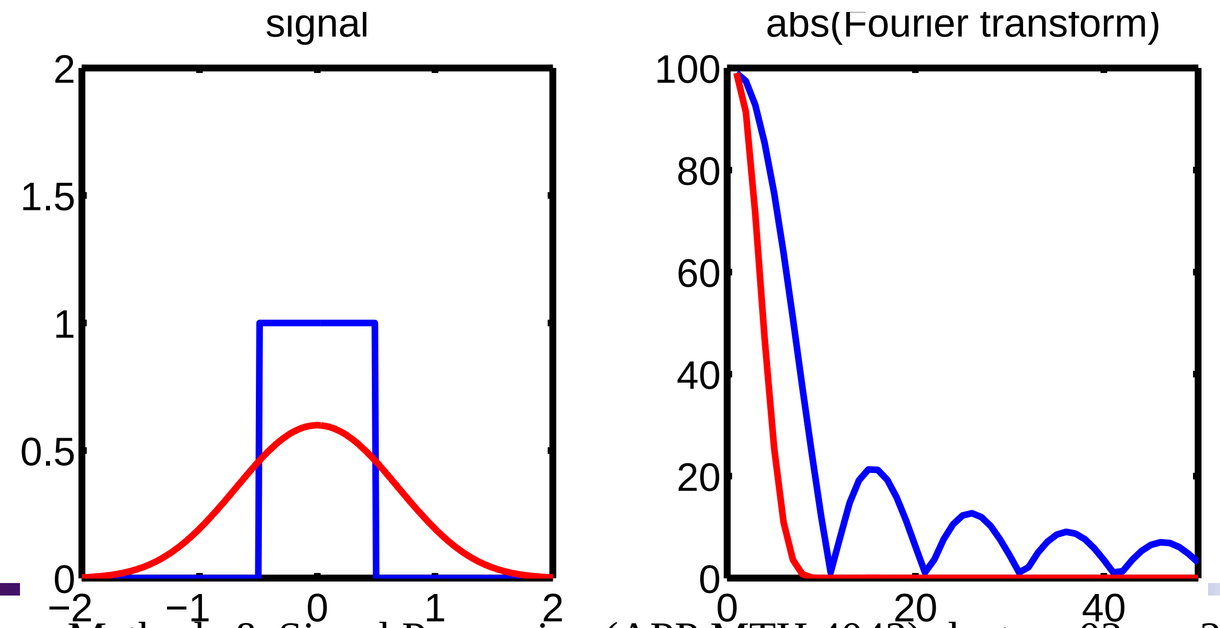


Limiting convolutions

$$\mathcal{F}\{r(t)\} = \text{sinc}(s) \Rightarrow \mathcal{F}\{r(t) * r(t) * \cdots * r(t)\} = \text{sinc}^n(s)$$

- n convolutions of a rectangular pulse produces a function with FT given by $\text{sinc}^n(x)$, which tends to a Gaussian as $n \to \infty$.
- The inverse FT of a Gaussian is also a Gaussian so the limit of $r(t) * r(t) * \cdots * r(t)$ is a Gaussian pulse.

4 convolutions

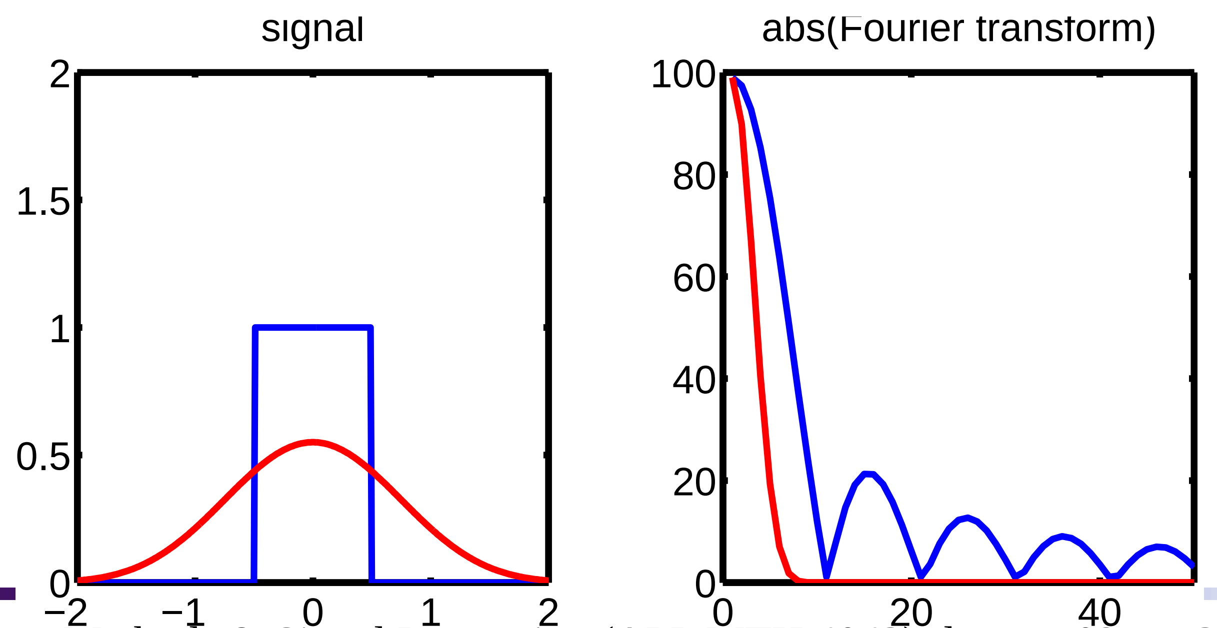


Limiting convolutions

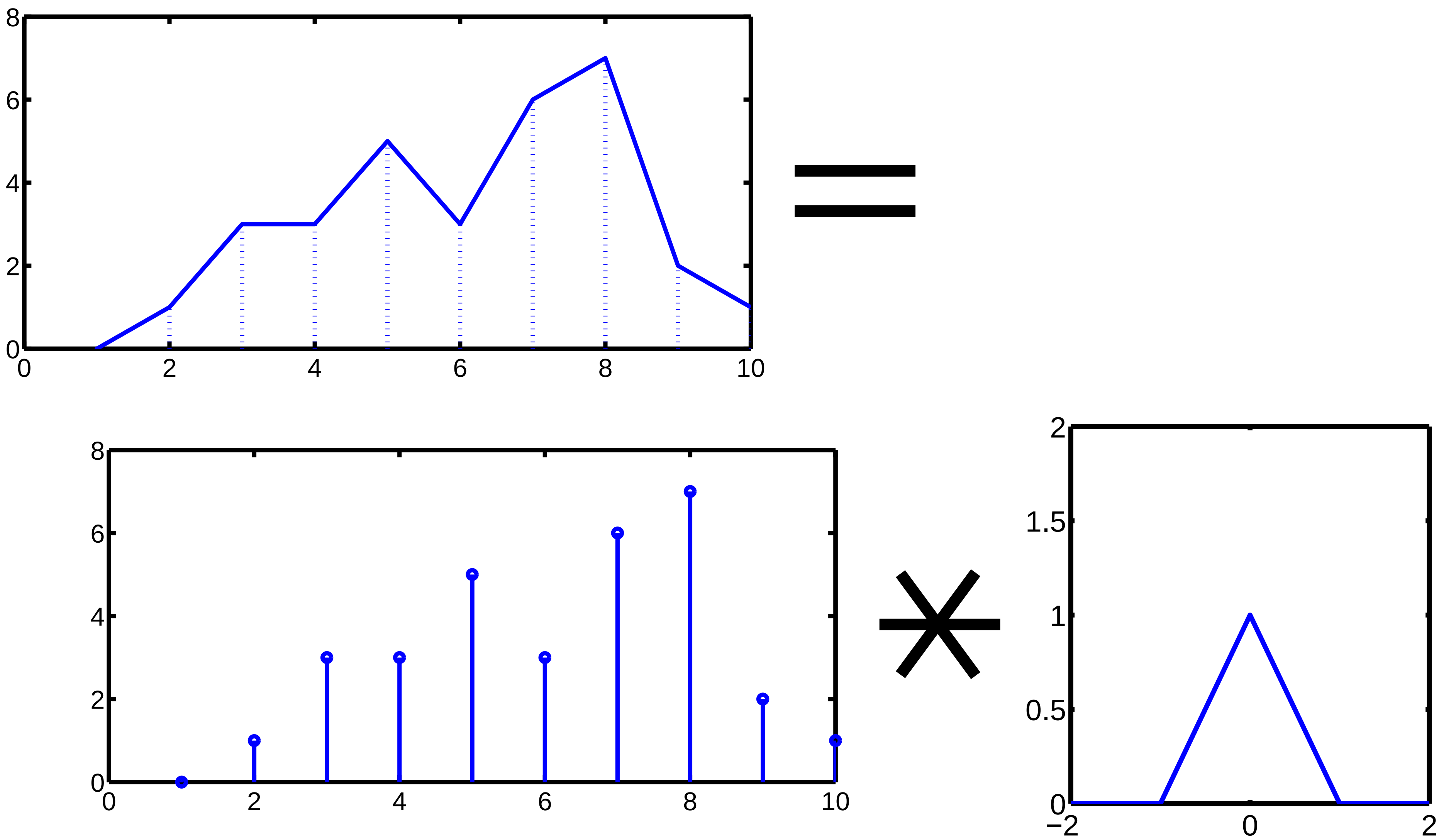
$$\mathcal{F}\{r(t)\} = \mathrm{sinc}(s) \Rightarrow \mathcal{F}\{r(t) * r(t) * \cdots * r(t)\} = \mathrm{sinc}^n(s)$$

- n convolutions of a rectangular pulse produces a function with FT given by $\mathrm{sinc}^n(x)$, which tends to a Gaussian as $n \to \infty$.
- The inverse FT of a Gaussian is also a Gaussian so the limit of $r(t) * r(t) * \cdots * r(t)$ is a Gaussian pulse.

5 convolutions



Convolution example: interpolation



Convolution example: interpolation

Fourier transformation of a piecewise linear function

$$f(t) = \left[\sum_{i=1}^{n} f_i \delta(t - t_i) \right] * r(t) * r(t)$$

is

$$F(s) = \left[\sum_{i=1}^{n} f_i e^{-i2\pi s t_i} \right] \mathrm{sinc}^2(s)$$

Properties: Diff. $\frac{d^n}{dt^n} f(t) \to (i2\pi s)^n F(s)$

$$\boxed{\mathcal{F}\left\{ \frac{d^n}{dt^n} f(t) \right\} = (i2\pi s)^n F(s)}$$

$$
\begin{aligned}
\mathcal{F}\left\{ \frac{d}{dt} f(t) \right\} &= \int_{-\infty}^{\infty} \frac{df}{dt} e^{-i2\pi st}\, dt \\
&= \int_{-\infty}^{\infty} \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t} e^{-i2\pi st}\, dt \\
&= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left[\int_{-\infty}^{\infty} f(t+\Delta t) e^{-i2\pi st}\, dt - \int_{-\infty}^{\infty} f(t) e^{-i2\pi st}\, dt \right] \\
&= \lim_{\Delta t \to 0} \frac{F(s) e^{i2\pi s\Delta t} - F(s)}{\Delta t} = F(s) \lim_{\Delta t \to 0} \frac{e^{i2\pi s\Delta t} - e^{i2\pi s0}}{\Delta t} \\
&= F(s) \left. \frac{d}{dt} e^{i2\pi st} \right|_{t=0} = i2\pi s F(s)
\end{aligned}
$$

and repeat (induction) for higher powers.

Properties: Differentiation II

$$\mathcal{F}\{(-i2\pi t)^n f(t)\} = \frac{d^n}{ds^n} F(s)$$

Similar to previous result,
but with respect to inverse Fourier transform.

Example: FT of a Gaussian

Another proof of the FT $G(s)$ of a Gaussian $g(t) = e^{-\pi t^2}$. Note that

$$g'(t) = -2\pi t g(t)$$

From the differentiation property

$$\mathcal{F}\{g'(t)\} = i2\pi s G(s)$$

From the dual differentiation property

$$\begin{aligned} \mathcal{F}\{-i2\pi t g(t)\} &= G'(s) \\ i\mathcal{F}\{g'(t)\} &= G'(s) \\ -2\pi s G(s) &= G'(s) \end{aligned}$$

Standard DE solutions give $G(s) = Ae^{-\pi s^2}$, and the constant $A = 1$ can be derived from the $s = 0$ term.

Some useful rules for FTs

$$
\begin{aligned}
F(-s) &= \int_{-\infty}^{\infty} f(t) e^{-i2\pi(-s)t}\, dt \\
&= \int_{-\infty}^{\infty} f(-t) e^{-i2\pi st}\, dt
\end{aligned}
$$

Evenness/Oddness of $F(s)$ is related to the properties of $f(t)$.

- even function $\Leftrightarrow$ even transform
- odd function $\Leftrightarrow$ odd transform

Some useful rules for FTs

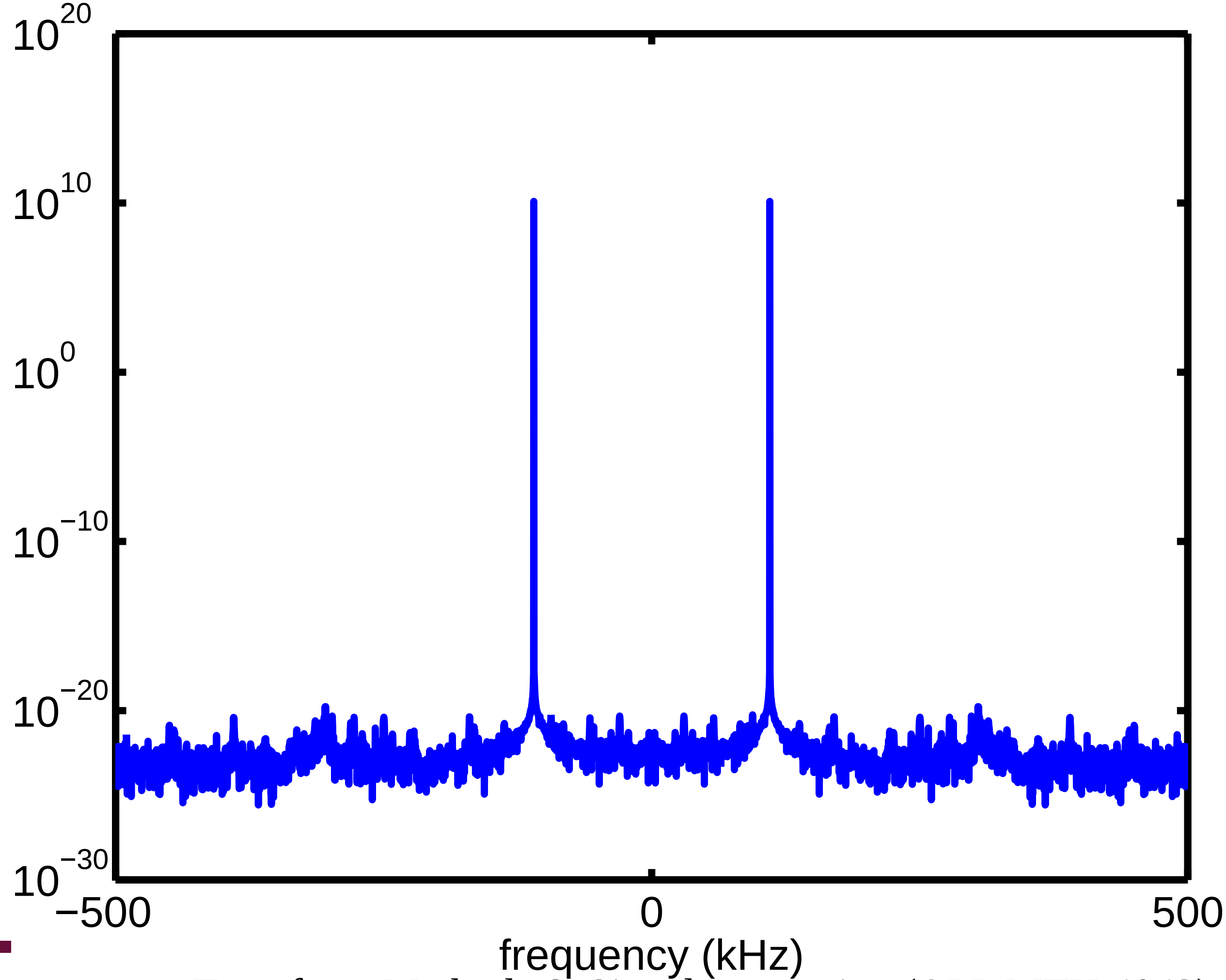
$$
\begin{aligned}
F^*(s) &= \int_{-\infty}^{\infty} f^*(t) e^{i2\pi st}\, dt \\
&= \int_{-\infty}^{\infty} f^*(t) e^{-i2\pi(-s)t}\, dt \\
&= \int_{-\infty}^{\infty} f^*(-t) e^{-i2\pi st}\, dt
\end{aligned}
$$

- real even function $\Leftrightarrow$ real even transform
- real odd function $\Leftrightarrow$ imaginary odd transform

Some useful rules for FTs

real even function ⇔ real even transform

Magnitude of Fourier transform of a cosine function.



Properties: Existence

Sufficient conditions

- $\int_{-\infty}^{\infty} |f(t)|\, dt$ exists
- There are a finite number of discontinuities in $f(\cdot)$
- $f(\cdot)$ has bounded variation

The Fourier transform exists for physical signals:
Some conditions above may be technically violated, e.g.

- DC current.
- infinite sin wave
- $\delta(x)$

For first two, can multiply by term like e^{-ax^2}, with small $a > 0$ to make integrals exist.

Properties: Invertible

If the conditions for existence are satisfied.

$$
\begin{aligned}
f(t) &= \mathcal{F}^{-1}\{\mathcal{F}\{f(t)\}\} \\
f(t) &= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(t) e^{-i2\pi st}\, dt \right] e^{i2\pi st}\, ds
\end{aligned}
$$

Where $f(t)$ is discontinuous, the equation should be replaced by

$$
\frac{1}{2}[f(t^+) + f(t^-)] = \mathcal{F}^{-1}\{\mathcal{F}\{f(t)\}\}
$$

Trigonometric basis

- similar to Fourier series: trigonometric functions used as a basis.
- here, can't assume fixed periodicity
- hence must include all sins and cosines
- think of $f(t)$ as containing a mix of periodic functions with different periods
- result is a continuous frequency spectrum

Measurement of spectra

The (continuous) Fourier transform allows us to examine mathematically the spectra of continuous functions, but is rarely useful in analyzing real signals. However, in some cases we can observe the spectra of real signals directly.

Measurement of Spectra

- how can we use the Fourier transform in practice?
- real signals are effectively continuous
  - sound waves are made of atoms
  - EM waves are made of photons
- how can we analyze frequencies?
  - we don't have an analytic function
  - we can't do the math directly

Measurement of Spectra

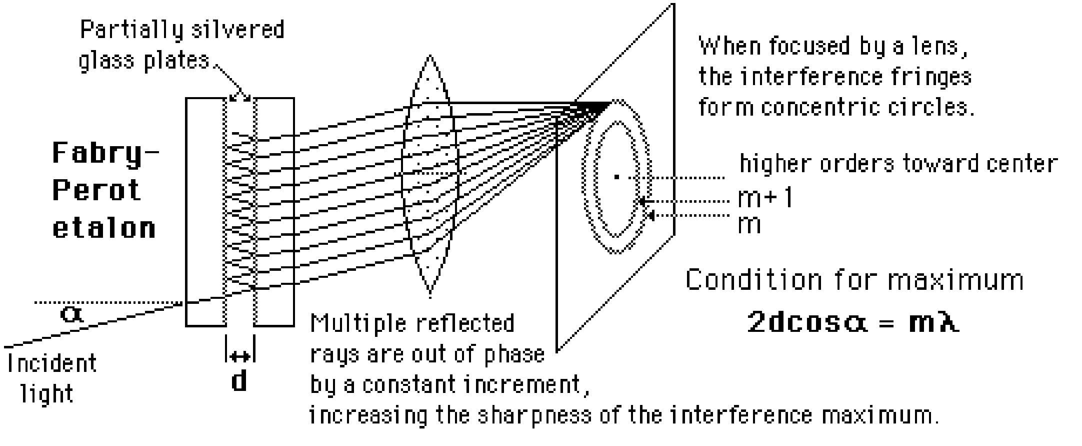
We can measure spectra directly in some cases

- radio frequencies, use a spectrum analyzer
- old ones are analogue
- think of as a bank of filters for each frequency
  - make copies of the signal
  - filter each copy for a particular frequency component
  - one filter per component you want to see

Measurement of Spectra

We can measure spectra directly in some cases

- light (can use massively parallel analogue devices)
  - prism
  - diffraction grating (a CD)
  - Fabry-Perot interferometer



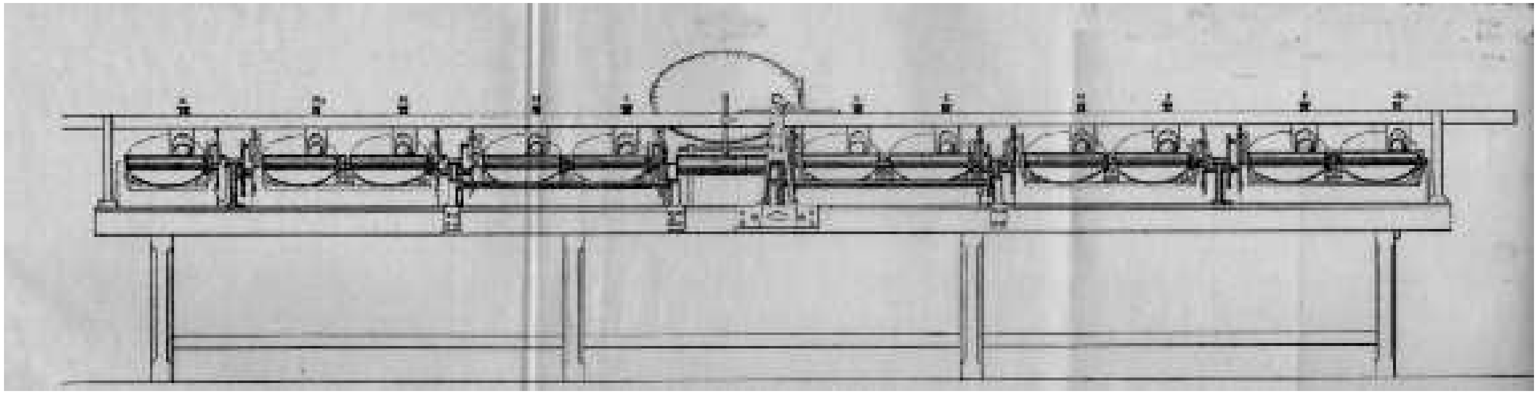
Measurement of Spectra

We can measure spectra directly in some cases

- tides, "The Harmonic Analyzer" Kelvin, analogues computation of coefficients of

$$A + B\sin t + C\cos t + D\sin 2t + E\cos 2t$$

The tidal gauge, tidal harmonic analyzer, and tide predictor, in Kelvin, Mathematical and Physical Papers (Volume VI), Cambridge 1911, pp 272-305.



http://www.math.sunysb.edu/~tony/tides/analysis.html

Measurement of Spectra

- The tidal gauge illustrates a point
- analogue devices
  - are hard to build
  - have limited resolution
  - are inflexible
- digital devices are often better
  - cheaper
  - more flexible
- we need to consider transforms of digital data
  - that's exactly what we'll do in the next lecture