Assignment 7: Solutions

TOTAL MARKS: 10

1. It is sufficient to show that the properties are true for an arbitrary element of the matrix.

[1 mark]

(a) Associativity: by the definition of $\otimes$

$$\begin{aligned} [A\hat{\otimes}(B\hat{\otimes}C)]_{ij} &= \bigoplus_{k=1}^{n} a_{ik} \otimes [(B\hat{\otimes}C)]_{kj} \\ &= \bigoplus_{k=1}^{n} a_{ik} \otimes \left(\bigoplus_{m=1}^{n} b_{km} \otimes c_{mj} \right) \\ &= \bigoplus_{k=1}^{n} \bigoplus_{m=1}^{n} a_{ik} \otimes b_{km} \otimes c_{mj}, \end{aligned}$$

by distributivity of $\otimes$ over $\oplus$, and where brackets have been removed from the internal products because $\otimes$ is associative. Then we can change the order of summation because $\oplus$ commutes, so

$$\begin{aligned} [A\hat{\otimes}(B\hat{\otimes}C)]_{ij} &= \bigoplus_{m=1}^{n} \bigoplus_{k=1}^{n} a_{ik} \otimes b_{km} \otimes c_{mj} \\ &= \bigoplus_{m=1}^{n} [(A\hat{\otimes}B)]_{im} \otimes c_{mj} \\ &= [(A\hat{\otimes}B)\hat{\otimes}C]_{ij}. \end{aligned}$$

[2 marks]

(b) Idempotence of $\hat{\otimes}$:

We could approach this by finding a counter-example. The operator min is idempotent, so consider the max-min (bottleneck) algebra.

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \hat{\otimes} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 3 & 4 \end{pmatrix} \neq \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

Another way to think about this would be that if $\hat{\otimes}$ were idempotent, then $A^2 = A$, and hence $A^k = A$ for all $k \geq 1$. Given that we would often choose idempotent $\oplus$ (which *does* imply idempotent $\hat{\oplus}$) we would then get that

$$A^* = I \hat{\oplus} A \hat{\oplus} A^2 \hat{\oplus} \cdots = I \hat{\oplus} A,$$

so if we do get this case, then it is rather unusual in that the algebra is 1-stable. That isn't true (for instance for the Bottleneck semiring), so we have a contradiction.

[2 marks]

2. (a)

$$A^* = \begin{pmatrix} 1.0 & 0.3 & 0.9 & 0.15 \\ 0.5 & 1.0 & 0.5 & 0.5 \\ 0.9 & 0.27 & 1.0 & 0.135 \\ 0.3 & 0.6 & 0.3 & 1.0 \end{pmatrix}$$

[3 marks]

(b) Each element A^*_{ij} gives the reliability of the most-reliable path between nodes i and j, in a graph whose link-reliabilities are given by A.
For instance $A^*_{12} = 0.3$ means that the most reliable path from 1 to 2 is the direct link between the two nodes.

[1 mark]

(c) The underlying graph has asymmetric weights (the reliabilities), and so is best thought of as a directed graphs, and the resulting most-reliable paths are therefore not all symmetric.

[1 mark]