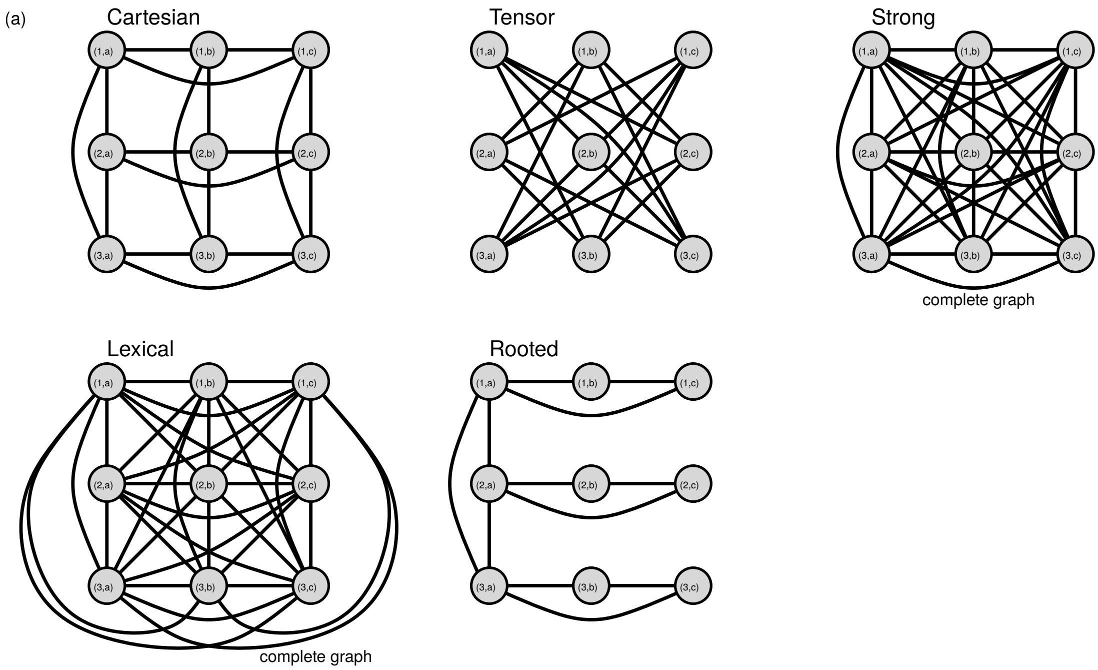
Assignment 5: Solutions

TOTAL MARKS: 4

1. Graph product solutions (only the Cartesian and rooted products are required this year).



[2 marks]

2. The graph is 4-regular and symmetric, *i.e.,* each node has degree 4, and it is symmetric in the sense that if we imagine the nodes on a torus, then each node connects to each of its neighbours. Thus the relationship between any pair of nodes in the same row or column is the same as any others, and the relationship between any pair of nodes diagonally adjacent is the same. Considering each of these possibilities in order consider Menger's Theorem, which says that

   > For an undirected graph G, the size of the minimum edge cut for an arbitrary pair of nodes $i \neq j$ is equal to the maximum number of edge-disjoint paths from i to j.

   - Nodes $(1, a)$ and $(1, b)$: they are connected by 4 edge-disjoint paths:
     - $(1, a) - (1, b)$
     - $(1, a) - (1, c) - (1, b)$
     - $(1, a) - (2, a) - (2, b) - (1, b)$
     - $(1, a) - (3, a) - (3, b) - (1, b)$
   - Nodes $(1, a)$ and $(2, b)$: they are connected by 4 edge-disjoint paths:
     - $(1, a) - (1, b) - (2, b)$
     - $(1, a) - (2, a) - (2, b)$
     - $(1, a) - (3, a) - (3, b) - (2, b)$
     - $(1, a) - (1, c) - (2, c) - (2, b)$

   We know they cannot be connected by 5 edge-disjoint paths because the node degree is only 4. Hence the network is 4-connected.

[2 marks]

3. Example code is included below:

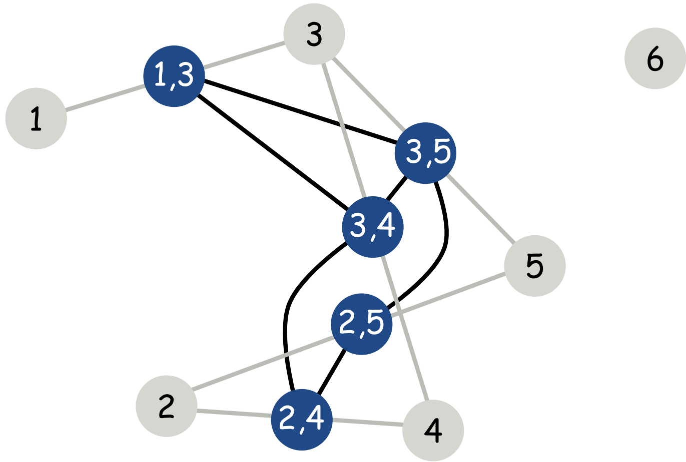
```
function A_L = line_graph(A_G)
%
% created:  Mon Aug 28 2017
% author:   M Roughan
% email:    matthew.roughan@adelaide.edu.au
%
% LINE_GRAPH:  determine the adjacency matrix of the line graph L(G)
%
% INPUTS:
%     A_g = nxn (symmetric) adjacency matrix of an input (undirected) graph G
%
% OUTPUTS:
%     A_L = |E(G)| x |E(G)| adjacency matrix of L(G)
%
assert(size(A_G,1) == size(A_G,2), 'A must be square');
assert(all(all(A_G == A_G')), 'graph is not undirected');
assert(min(min(A_G)) >= 0, 'adjacency matrices must be non-negative');

% number of nodes and edges n=|N(G)|, and e=|E(G)|
n = size(A_G,1);
e = sum(sum(A_G))/2;

% output graph will be |E(G)| x |E(G)|
A_L = zeros(e, e);
% list edges in lexicographic order
edges = zeros(e,2);
count = 1;
for i=1:n
  for j=i+1:n
    if A_G(i,j)==1
      edges(count,:) = [i,j];
      count = count + 1;
    end
  end
end
assert(e == count-1, 'something went wrong here');

% do this the crude but obvious way, node by node
for i=1:n
  % for each node connected to i, there will be a corresponding set of edges incident
  k = find(A_G(i,:) > 0);
  if ~isempty(k)
    m = find( (edges(:,1) == i & ismember(edges(:,2),k)) | ...
      (edges(:,2) == i & ismember(edges(:,1),k)) );
    A_L(m,m) = 1; % create a small clique of these edges
  end
end
A_L = A_L - diag(diag(A_L)); % get rid of diagonal elements
```

The line graph of the problem set is shown below:



Its adjacency matrix is

$$A = \begin{array}{c} \\ (1,3) \\ (2,4) \\ (2,5) \\ (3,4) \\ (3,5) \end{array} \begin{array}{c} \begin{array}{ccccc} (1,3) & (2,4) & (2,5) & (3,4) & (3,5) \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array} \tag{1}$$

and the degrees of its nodes are shown below.

node of $L(G)$	degree
(1,3)	2
(2,4)	2
(2,5)	2
(3,4)	3
(3,5)	3

4. **[Mathematical derivation]**

A directed ring graph will have nodes with in- and out-degree equal to exactly 1 for all nodes.

Thus the adjacency matrix will have exactly 1 "1" in each row and column. In particular it will be a Toeplitz matrix with 1 along the diagonal above the main diagonal, a 1 in the bottom left corner, and zeros everywhere else, *e.g.*, for $n = 4$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

This is an example of a permutation matrix P. Products (and hence powers P^i) of permutation matrices are all permutation matrices (members of the permutation group)[1].

In particular, it is a *circulant* matrix, which is a special type of Toeplitz matrix where each row is rotated one element to the right of the preceding row, so we are dealing with *cyclic permutations*, that shifts nodes by some offset (returning to the beginning). Repeating it is equivalent to multiplication, or shifting an additional time, or operations on the cyclic group.

[1]A permutation matrix can be thought of as a relabelling, and so the new network resulting from a power of the matrix will still be isomorphic to a direct ring graph.

Hence, for a network with n nodes, n+1 shifts will bring you back to a shift of 1, and hence back to the original adjacency matrix.

[**Alt. Intuitive derivation**]

Powers A^k of the adjacency matrix give the number of paths of length exactly k.

Consider the ring graph, there is only ever one path of length k from any one node to any other, and that will go to a node k hops around the ring. Thus the matrices A^k are still 0-1 Toeplitz matrices, with exactly one zero in each row and column.

When we reach $k = n$, we have returned back to the original node. Thus $k = n + 1$ will correspond to a path back to the node adjacent to the original, and hence $A^{k+1} = A$.