Information Theory and Networks

Lecture 14: Practical Compression

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Part I

Practical Compression

Baseball is 90 percent mental and the other half is physical.
Yogi Berra

Section 1

Asymptotic Equipartition Property (AEP)

Weak Law of Large Numbers

For independent, identically distributed (IID) RVs X_i, then as $n \to \infty$

$$\frac{1}{n} \sum_{i=1}^{n} X_i \xrightarrow{p} E[X_i]$$

where convergence is in probability.

AEP

- Uses the Law of Large Numbers to find an approximation for entropy in terms we can realize from observed sequences
- Flipping it around, probabilities of observed sequences of n symbols will be close to 2^{-nH}
  - almost all events are equally surprising
- Allows division of possible sequences into
  - typical
  - non-typical

  Properties proved for typical set will be true with high probability.

AEP formalized

Theorem (AEP)

If $X_1, X_2, \ldots$ are IID with PMF $p(x)$, then

$$-\frac{1}{n} \log P(x_1, x_2, \ldots, x_n) \xrightarrow{p} H(X)$$

Proof.

Functions of independent RVs are also independent RVs, so the $P(X_i)$ and $\log P(X_i)$ are IID RVs, so

$$\frac{1}{n} \log P(x_1, x_2, \ldots, x_n) = \frac{1}{n} \log \prod_{i=1}^{n} p(x_i) = \frac{1}{n} \sum_{i=1}^{n} \log p(x_i).$$

Hence, by the Weak Law of Large Numbers:

$$-\frac{1}{n} \log P(x_1, x_2, \ldots, x_n) \xrightarrow{p} -E\left[\log p(X)\right] = H(X).$$

AEP intepretation

- So in the limit

$$-\frac{1}{n} \log P(x_1, x_2, \ldots, x_n)$$

  is close to $H(X)$
- Or $P(x_1, x_2, \ldots, x_n)$ is typically close to

$$2^{-nH(X)}$$

  (remembering we take logs to base 2 in the default definition of entropy)

Typical Sequences

Definition (typical)

The typical set $A_\epsilon^{(n)}$ with respect to the PMF $p(x)$ is the set of sequences $(x_1, x_2, \ldots, x_n) \in \Omega^n$ with the property

$$2^{-n(H(X)+\epsilon)} \leq P(x_1, x_2, \ldots, x_n) \leq 2^{-n(H(X)-\epsilon)}.$$

Properties:

1. $P\left(A_\epsilon^{(n)}\right) > 1 - \epsilon$ for sufficiently large n.
   (follows directly from the AEP theorem)
2. $\left|A_\epsilon^{(n)}\right| \leq 2^{n(H(X)+\epsilon)}$
   Proof [CT91, Chapter 3, p.52]
3. for other properties see [CT91, Chapter 3, p.52]

Consequences for compression

1. We can divide the set of possible sequences into
   1. typical $A_\epsilon^{(n)}$
   2. atypical $\Omega^n \backslash A_\epsilon^{(n)}$
2. For sufficiently long sequences, the typical set is both
   1. very likely
   2. relatively small, compared to all possible sequences, if the entropy is small
3. It suggests a compression method
   1. For typical sequences
      1. Assign, in any order you like, a number to each sequence
      2. The code is just this number, in binary, prefixed by zero
   2. For atypical sequences, assign them a number too
      1. Assign, in any order you like, a number to each sequence
      2. The code is just this number, in binary, prefixed by one

Consequences for compression

1. It suggests a compression method
   1. For typical sequences the code is has binary length, at most
      $$\ell = n(H + \epsilon) + 1 + 1$$
      1. There are less than $2^{n(H+\epsilon)}$ sequences, so we need numbers with $n(H + \epsilon)$ bits.
      2. The first $+1$ arise from prefixing with a zero
      3. The second $+1$ arise because $n(H + \epsilon)$ might not be an integer
   2. For atypical sequences the code is has binary length, at most
      $$\ell = n \log_2 |\Omega| + 1 + 1$$
      1. The first $+1$ arise from prefixing with a one
      2. The second $+1$ arise because $n \log_2 |\Omega|$ might not be an integer

Consequences for compression

Theorem (Expected Message Length)

If $X_1, X_2, \ldots$ are IID with PMF $p(x)$, then for any $\epsilon' > 0$, there exists a code which maps sequences of length n into binary strings such that the mapping is one-to-one) and therefore invertible and

$$E\left[\frac{1}{n}\ell(X_1, X_2, \ldots, X_n)\right] \leq H(X) + \epsilon',$$

for n sufficiently large.

Consequences for compression

Proof.

Use the coding method described above, then

$$
\begin{aligned}
E\left[\ell(\mathbf{x})\right] &\leq \sum_{\mathbf{x}} p(\mathbf{x})\ell(\mathbf{x}) \\
&= \sum_{\mathbf{x}\in A_\epsilon^{(n)}} p(\mathbf{x})\ell(\mathbf{x}) + \sum_{\mathbf{x}\notin A_\epsilon^{(n)}} p(\mathbf{x})\ell(\mathbf{x}) \\
&\leq \sum_{\mathbf{x}\in A_\epsilon^{(n)}} p(\mathbf{x})\left[n(H+\epsilon)+2\right] + \sum_{\mathbf{x}\notin A_\epsilon^{(n)}} p(\mathbf{x})\left[n\log|\Omega|+2\right] \\
&= P\left(A_\epsilon^{(n)}\right)\left[n(H+\epsilon)+2\right] + \left(1-P\left(A_\epsilon^{(n)}\right)\right)\left[n\log|\Omega|+2\right] \\
&\leq n(H+\epsilon)+\epsilon n\log|\Omega|+2
\end{aligned}
$$

Which satisfies the theorem if we take $\epsilon' = \epsilon + \epsilon\log|\Omega| + 2/n$, because that can be made arbitrarily small for suitable choice of ϵ and n. □

Consequences for compression

Corollary

Don't code per symbol!

- The above gives us a bound on coding of $H(X)$ bits per symbol in the original sequence.
- Simple counter example:
  - Sequence
$$aaaaaaaaaaaaaaaaaaaa$$
  - Has $P(a) = 1$, and $H(X) = 0$.
  - Best coding per symbol still needs one bit per symbol, e.g., it isn't close to the best coding
  - Better: run-length coding
$$aaaaaaaaaaaaaaaaaaaa \leftrightarrow 20'a's$$
- So now we are considering new n-length symbols

Section 2

Some Compression algorithms

Run length encoding (RLE)

If our data has many sequences of the same symbol

- record the symbols, and how long each run is, so

$$aaaaabbbbbaaaaaaabbbbbaaaaaaaaabbbbb$$

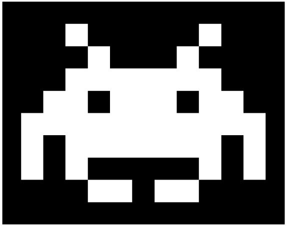
  becomes

$$5a5b7a5b9a5b$$

- 36 symbols becomes 12
  - "alphabet" may be bigger though, as now we include numbers
- Compression factor depends on the data, a lot.

Run length encoding (RLE)

Use for instance in bitmapped images, with a limited palette:



- directly encoded: $10 \times 13 = 130$ bits

  `0000000000000111000001100001011111000110110100011110100011110000011110`

- run length encoded: 38 numbers

  `15,3,6,2,5,1,1,5,4,2,1,2,1,1,4,4,1,1,4,4,6,4,1,1,3,2,1,2,1,1,2,1,1,5,6`

  but if we just record the numbers
  - 8 bits then code $= 38 \times 8 = 304$ bits
  - 4 bits (minimal) $= 38 \times 4 = 152$ bits

Run length encoding (RLE)

- Run length encoded: 38 numbers
  `15,3,6,2,5,1,1,5,4,2,1,2,1,1,4,4,1,1,4,4,6,4,1,1,3,2,1,2,1,1,2,1,1,5,6`
  but if we just record the numbers
  - 8 bits then code $= 38 \times 8 = 304$ bits
  - 4 bits (minimal) $= 38 \times 4 = 152$ bits
- What if we Huffman encode the numbers?

$$H(X) \simeq 2.54$$

So the total number of bits (assuming efficient encoding) would be

$$38 \times 2.54 \simeq 97 \text{ bits}$$

which is slightly better than 130 bits for the raw file.
- Compare Huffman coding of original with blocks of 5 gives about 73 bits, so we may as well just do a raw Huffman code.

Further reading I

Thomas M. Cover and Joy A. Thomas, *Elements of information theory*, John Wiley and Sons, 1991.

Raymond W. Yeung, *Information theory and network coding*, Springer, 2010.