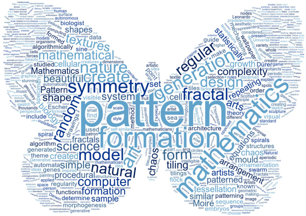
Advanced Mathematical Perspectives 1

Lecture 20: Cellular Automata



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www.maths.adelaide.edu.au/matthew.roughan/notes/AMP1/

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THE UNIVERSITY of ADELAIDE

SUB CRUCE LUMEN

ACEMS

AUSTRALIAN RESEARCH COUNCIL CENTRE OF EXCELLENCE FOR MATHEMATICAL AND STATISTICAL FRONTIERS

I used to feel guilty in Cambridge that I spent all day playing games, while I was supposed to be doing mathematics. Then, when I discovered surreal numbers, I realized that playing games IS math.

John Horton Conway

All the wonders of our universe can in effect be captured by simple rules, yet ... there can be no way to know all the consequences of these rules, except in effect just to watch and see how they unfold.

Stephen Wolfram

"And Wolfram knows about cellular automata? Oh, my goodness, yes," said Anna. "He wrote a book you could kill a man with ..."

Robert J. Sawyer

Section 1

Complexity

What is complexity?

There are lots of definitions of complexity. I would like to distinguish between *complex* and *complicated*

- Complicated: lots of moving parts, but governed by known, predictable laws.
  A gas is complicated – its has many, many molecules, and their detailed interactions are very complicated, but we can describe it simple using statistical representations. The math might be sophisticated, and the problem hard in many ways, but we can do good predictions. A clock is another good example.
- Complex: intrinsically hard to understand or predict.
  Human inter-personal interactions are complex. While aspects might be described by mathematical laws, these are always gross approximations, and usually proved by the exceptions. And prediction is hard.

Sometimes the difference might be how smart we are, but there seems to be a difference in category not just quantity.

We are interested in *complex* systems.

Ex nihilo nihil fit: Nothing comes out of nothing

Ex nihilo nihil fit: Nothing comes out of nothing

therefore

Complexity must come from complexity????

Section 2

Conway's Game of Life

Conway's Game of Life [Gar70]

- Define a grid of square "cells"
  - each is "alive" or "dead"
- At each time step, we count the number of each cell's neighbours that are alive, and
  1. any live cell with fewer than 2 live neighbours dies
  2. any live cell with more than three neighbours dies
  3. any live cell with 2 or 3 live neighbours lives on
  4. any dead cell with exactly three live neighbours becomes alive, and otherwise stays dead

Mathematical description

$$x_{ij}^{(t)} = \begin{cases} 1, & \text{if cell } (i,j) \text{ is alive at time } t \\ 0, & \text{otherwise} \end{cases}$$

Neighbourhood of (i,j) is a set, which for Conway's game is defined as

$$\begin{aligned} N_{i,j} \quad = \{ \quad & (i-1,j-1),(i-1,j),(i-1,j+1), \\ & (i,j-1),(i,j+1), \\ & (i+1,j-1),(i+1,j),(i+1,j+1)\} \end{aligned}$$

The next step is based on the number of alive neighbours given by

$$n_{ij}^{(t)} = \sum_{(k,m)\in N_{i,j}} x_{km}^{(t)}$$

Mathematical description

Rules:

- If $x_{ij}^{(t)} = 1$, *i.e.*, the cell is alive

$$x_{ij}^{(t+1)} = \begin{cases} 1, & \text{if } T_{min}^{alive} \leq n_{ij}^{(t)} \leq T_{max}^{alive} \\ 0, & \text{otherwise} \end{cases}$$

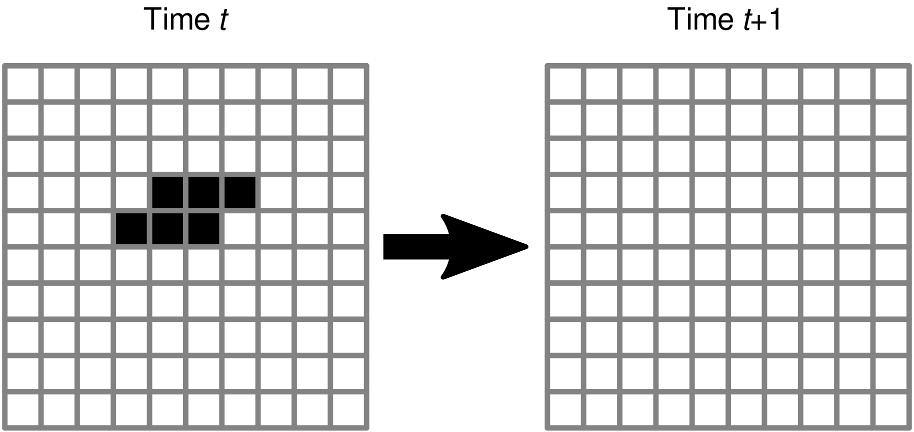
- If $x_{ij}^{(t)} = 0$, *i.e.*, the cell is dead

$$x_{ij}^{(t+1)} = \begin{cases} 1, & \text{if } T_{min}^{dead} \leq n_{ij}^{(t)} \leq T_{max}^{dead} \\ 0, & \text{otherwise} \end{cases}$$

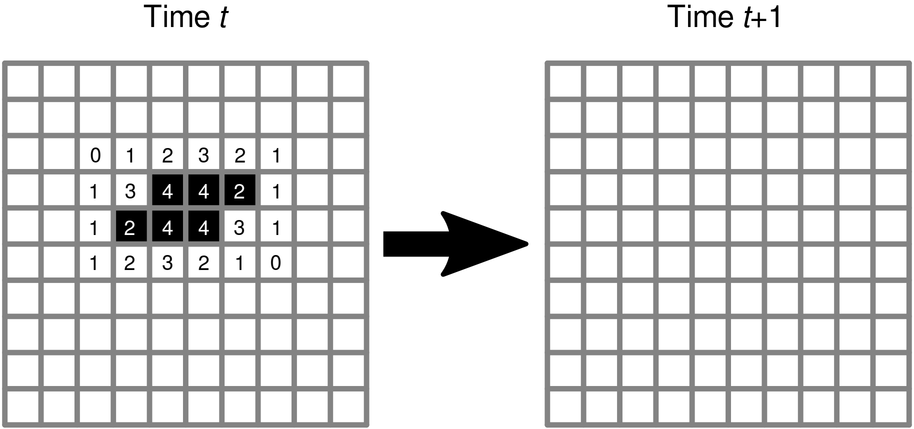
In Conway's game

$$\begin{aligned} T_{min}^{alive} &= 2 \\ T_{max}^{alive} &= 3 \\ T_{min}^{dead} &= 3 \\ T_{max}^{dead} &= 3 \end{aligned}$$

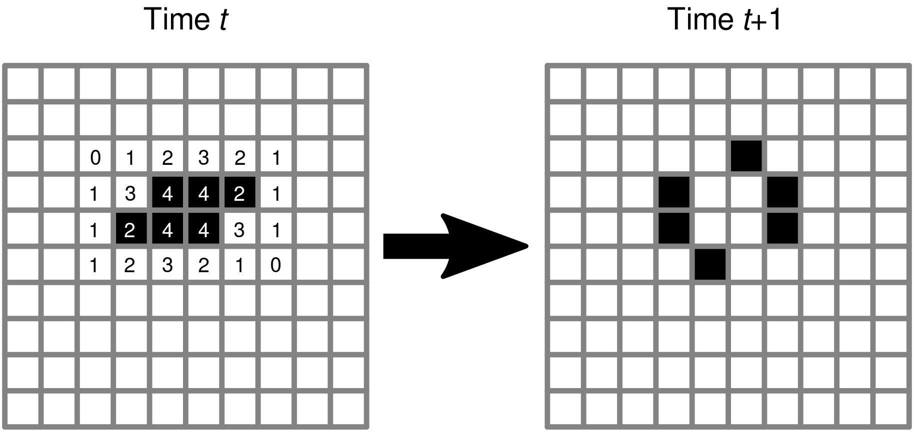
Example



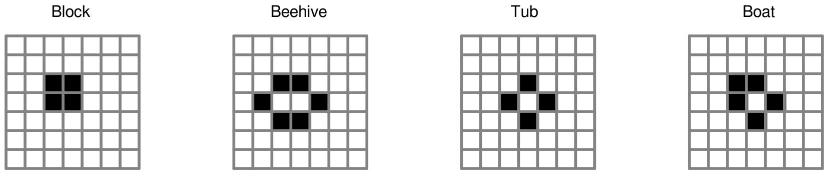
Example



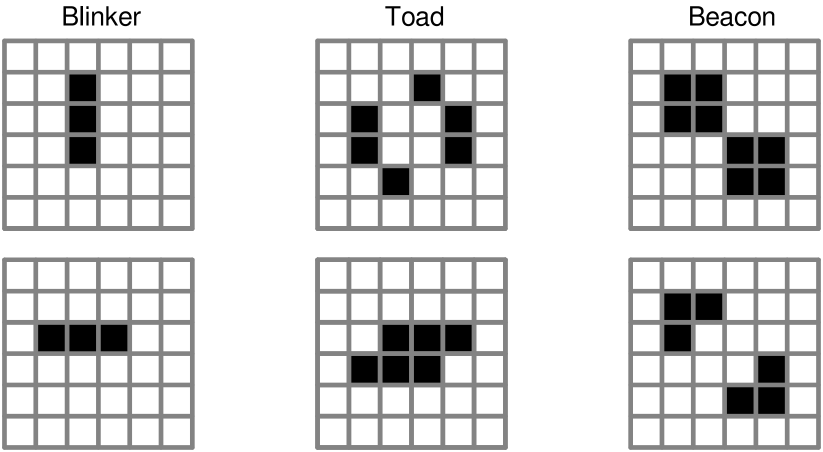
Example



Example: Stable Patterns

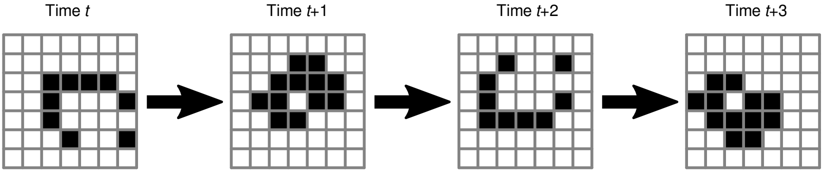


Example: Simple Oscillators (period 2)



Example: Lightweight Spaceship

- This pattern is a simple *glider* or spaceship



  - so the patterns repeats, but shifted to the left by 2 cells

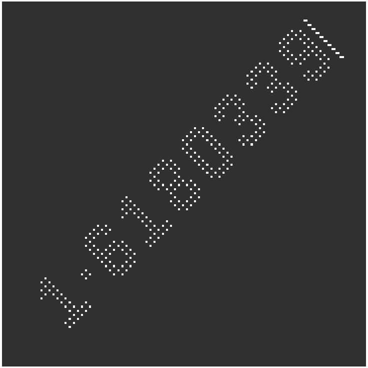
- Let's have a look at Golly
  http://golly.sourceforge.net/

http://www.conwaylife.com/w/index.php?title=Spaceship

Richness of the space

- Period n oscillators
- Larger spaceship
- Glider guns
- Puffers (like spaceships but which leave debris)
- Replicators
-

GoL Example: Golden ratio φ



Large-scale view (you can't see the individual cells here) generated with Golly and `http://pentadecathlon.com/lifeNews/2011/01/phi.mc`, *e.g.*, see `http://pentadecathlon.com/lifeNews/2011/01/phi_and_pi_calculators.html`

There is a similar one for π, *e.g.*, see `http://pentadecathlon.com/lifeNews/2011/01/pi.mc`

GoL = Turing Complete

Definition

A computer programming language (or system, or automata) is Turing complete if it can be used to simulate any Turing machine.

If so, then it can compute anything computable by a real-world computer.

The GoL is Turing complete (assuming the grid is infinite[1]).

- Glider's can represent a stream of bits
- Logic gates can be built from intersections of streams

In fact, it's been built [Ren11]

[1] This is a limitation of all real-world computers.

Mathematics, Life and Art

The GoL inspired

- musician Brian Eno to create generative music
  https://www.youtube.com/watch?v=UqzVSvqXJYg
  https://www.youtube.com/watch?v=IGUEVXqvCwM
  http://longnow.org/seminars/02006/jun/26/playing-with-time/
- Will Wright – creator of Sim City
- Cellular Automata in MIDI music [BELM04]
- Robert Bosch http://www.dominoartwork.com/life.html
- Francis Bitonti fashion designer
  https://www.wired.com/2014/10/3-d-printed-shoes-generated-using-conways-game-life/

BTW: Clock of the long now http://longnow.org/clock/

Generalisations

Konrad Zuse proposed that the physical laws of the universe (at the lowest level) are some variant of cellular automata, but not Conway's. So what can we tweak?

- Neighbourhood — *e.g.*, could be vertical and horizontal, or include diagonals, or could include cells more than one step away.
- Shape of grid could be based on any tessellation — *e.g.*, could be hexagonal.
- Grid can be finite or infinite, or toroidal (periodic), or dynamically growing
- 1D or 2D or 3D or even non-Euclidean spaces
- Number of states, not just alive or dead
- The most important bit (maybe) the "RULES"
  - counts (for alive and dead)
  - symmetry

Are there some universal properties in all of these variants?

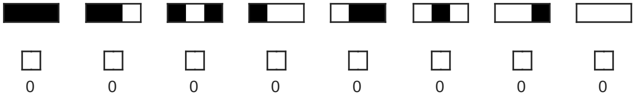
Section 3

Wolfram and Rule 30

Wolfram studied a set of cellular automata (in 1D)

1. In 1D, rules are given by their pattern
   - the rule relates three parents to the child
   - all based on binary codes

Rule 0:



2. Each cell has 2 neighbours + its own state, so there are 3 states, each with two possibilities so $2^3 = 8$ possible states (see above), and hence $2^8 = 256$ possible rules.

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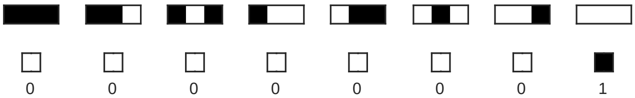

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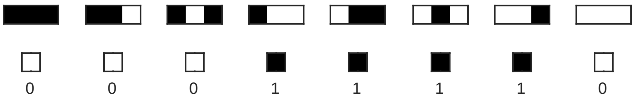

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Examples

- Start from a row with a single alive cell
- Each row is one step in time

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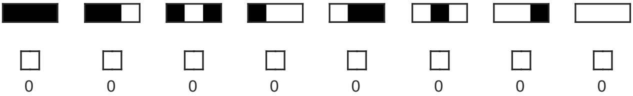

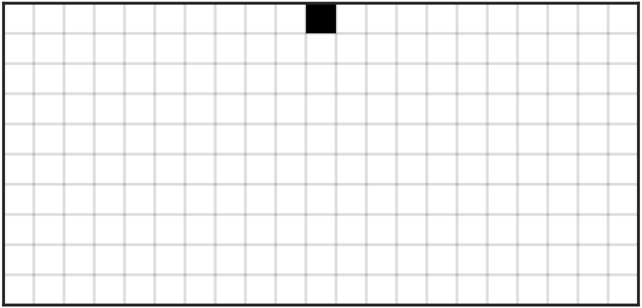

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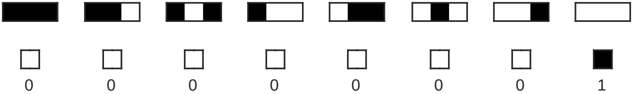

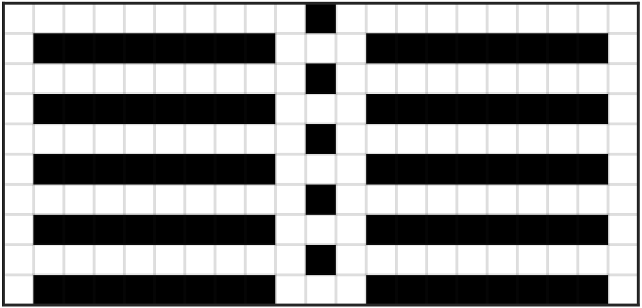

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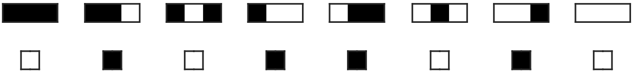
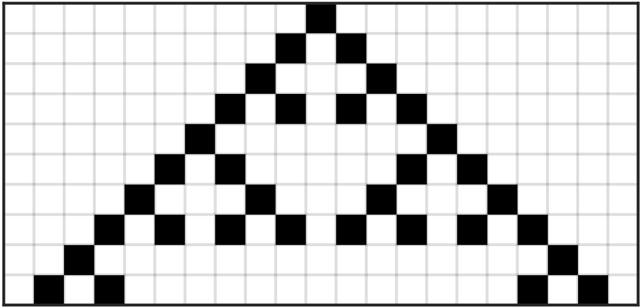
Rule 90:



0 1 0 1 1 0 1 0



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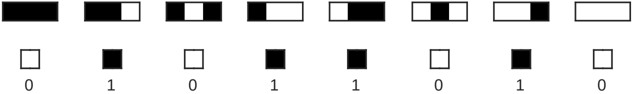

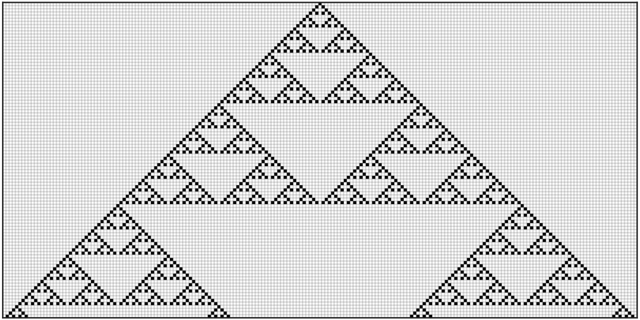

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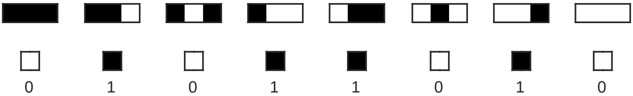
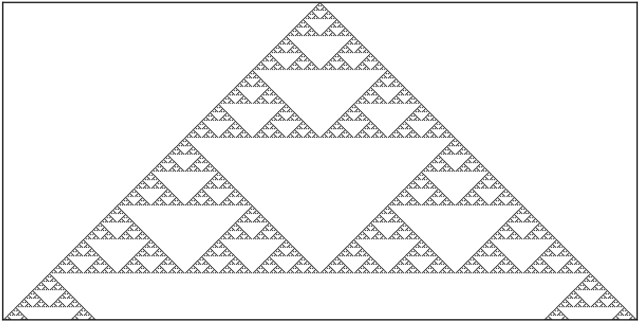




Wolfram grouped the rules into classes by the criteria:

1. Nearly all initial patterns evolve quickly into a stable, homogeneous state.
2. Nearly all initial patterns evolve quickly into stable or oscillating structures.
3. Nearly all initial patterns evolve in a pseudo-random or chaotic manner. Any stable structures that appear are quickly destroyed by the surrounding noise.
4. Nearly all initial patterns evolve into structures that interact in complex and interesting ways, with the formation of local structures that are able to survive for long periods of time.
   Class 2 type stable or oscillating structures may be the eventual outcome, but the number of steps required to reach this state may be very large, even when the initial pattern is relatively simple.

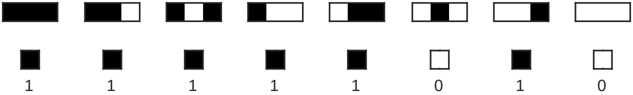
Conway's Game of Life is in the last class. Wolfram's most famous example he called Rule 30, and is also in this class.

Effect of Initial State and Locality

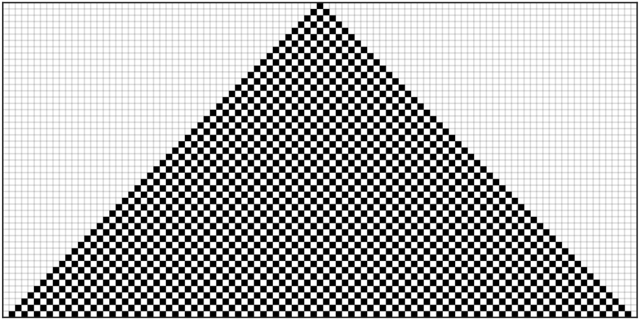
1. Any randomness in the initial pattern disappears.
2. Some of the randomness in the initial pattern may filter out, but some remains. However, local changes to the initial pattern tend to remain local.
3. Local changes to the initial pattern tend to spread indefinitely.
4. Local changes to the initial pattern may spread indefinitely.

Example: Class 1: *e.g.*, Rules 250 and 254

Rule 250:

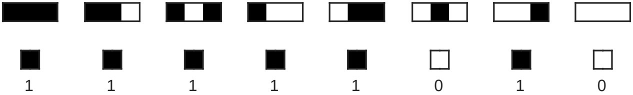


Start with a single alive cell

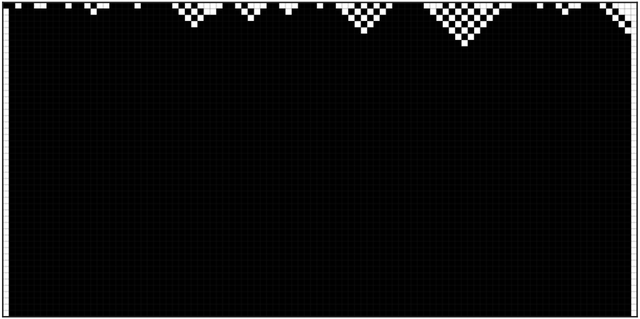


Example: Class 1: *e.g.*, Rules 250 and 254

Rule 250:

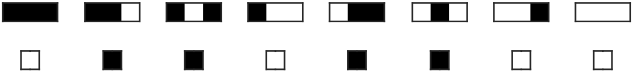


Start with random cells



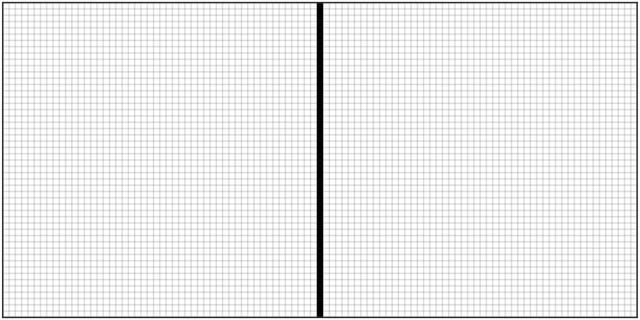
Example: Class 2: *e.g.*, Rules 4 and 108

Rule 108:



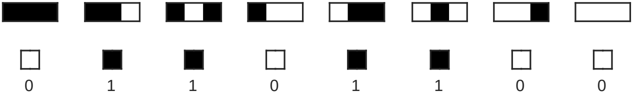
0 1 1 0 1 1 0 0

Start with a single alive cell

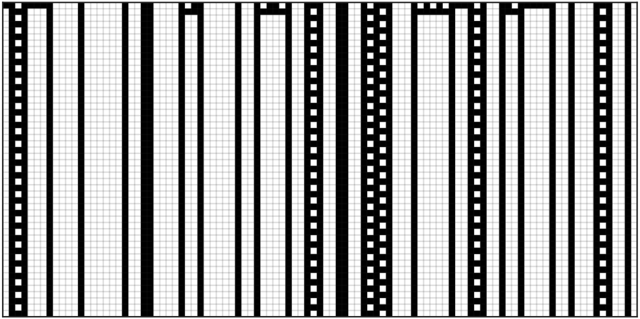


Example: Class 2: *e.g.*, Rules 4 and 108

Rule 108:

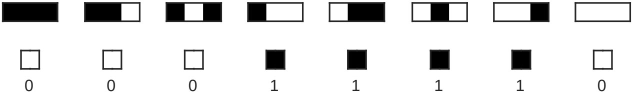


Start with random cells

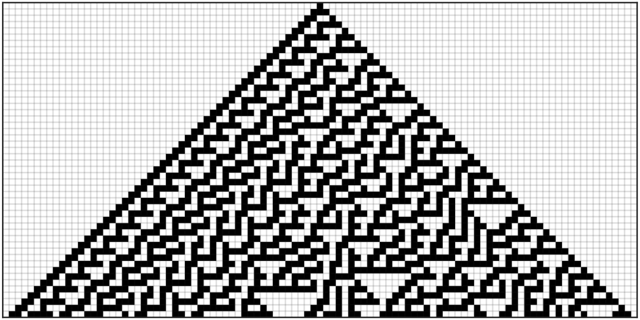


Example: Class 3: *e.g.*, Rules **30** and 90

Rule 30:



Start with a single alive cell



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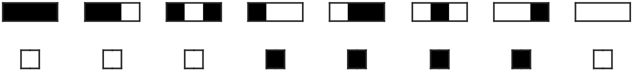
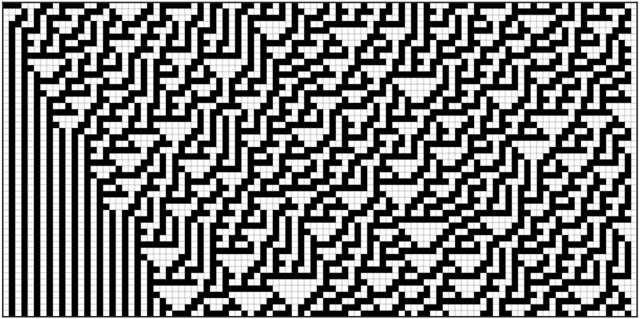


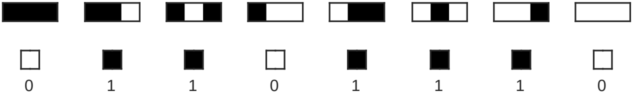
0 0 0 1 1 1 1 0

Start with random cells

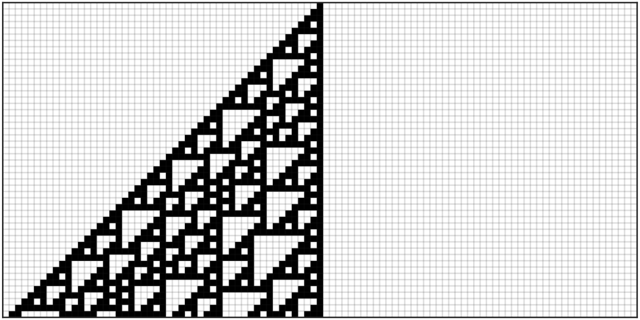


Example: Class 4: Rule 54 and 110

Rule 110:

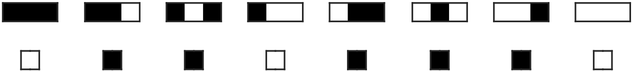


Start with a single alive cell



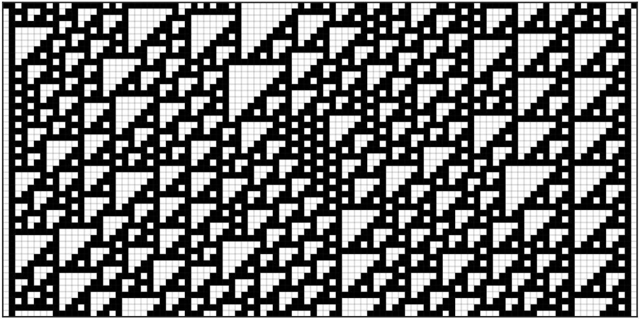
Example: Class 4: Rule 54 and 110

Rule 110:



0 1 1 0 1 1 1 0

Start with random cells



Connections

- Pseudo-Random Numbers can be seen as a special type of Class 3 cellular automata
  - the pattern is deterministic, but so complicated the only way to predict it is to run the automata, so we can treat its outcome as almost random
- Self-organisation
  - some systems seem to "self-organise," which seems to be a contradiction with the laws of thermodynamics (specifically the second, *i.e.,* disorder increases)
  - a parallel is seen in life
- Computation
  - links to ideas of Turing complete computer systems

Wolfram's ideas [Wol02]

Wolfram's ideas (and others in this area) parallel and motivate some of the themese that run through this course

- Relationships between discrete and continuous models
- Complexity out of simple rules
- Apparent randomness, despite simple structure
- The "edge of chaos" – natural patterns often form in between the extremes of boring regularity and complete randomness

Mathematica and α

BTW Wolfram α

- Came out of Mathematica (the thing Wolfram is best known for)
- Symbolic Manipulation + Computer Algebra
- Lots of alternatives now
  https://en.wikipedia.org/wiki/List_of_computer_algebra_systems
- You should try, at least, Wolfram α

Section 4

Swarms and murmurations

- How far can we take the idea of simple, local rules producing interesting complexity?
- Do we see this type of thing in nature?
- How does it fit into the scheme of models we have already looked at?

	Discrete	Continuous
Random (Stochastic)	Random Walk(s)	*???*
Deterministic	Difference approximation, Cellular automata[1]	Reaction-Diffusion

Table: Model classes.

1 - Note: difference approximations are built on a discrete grid, but have continuous state, whereas cellular automata have a discrete state as well.

A murmuration of swallows

https://www.youtube.com/watch?v=V4f_1_r80RY

- A *murmuration* is the collective name for a flock of starlings, the term probably derived from the sound it generates
- How do flocks of birds form large patterns as they fly?
  - no-one is planning it
  - there is no central authority to tell them what to do

Other examples: https://www.youtube.com/watch?v=NREmtGhIHew
https://www.youtube.com/watch?v=QOGCSBh3kmM

Ants and Swarms

The Ants toil for no Master
Sufficient to their Need
The daily commerce of the Nest
The storage of their Seed
They meet-and exchange Messages-
But none to none-bows down
They-like God's thoughts-speak to each to each-
Without-external-crown

A.S.Byatt, Possession

Ants and Swarms [Gor99]

- Why do ants bring food home to the nest walk in (somewhat) straight lines?
  - individual ants are stupid
  - they can't see very far
  - they don't know Euclidean geometry
- How do swarms insects create hives?

Fish and Schools

- How do fish form into schools
- How is complex behaviour orchestrated in the school

https://en.wikipedia.org/wiki/Shoaling_and_schooling

No answers today – go away and find out.

Takeaways

- Complexity can arise from simple rules
- Long-range structure can arise from local rules
- Tiny changes can result in big differences in later states, so we can only see final outcome by tracing the system forward
  - more on this later

Section 5

Extras

Links

- http://bactra.org/notebooks/cellular-automata.html
- https://plato.stanford.edu/entries/cellular-automata/index.html
- https://plato.stanford.edu/entries/cellular-automata/supplement.html
- http://golly.sourceforge.net/
- https://www.youtube.com/watch?v=UqzVSvqXJYg
- http://www.math.cornell.edu/~lipa/mec/lesson6.html
- http://pentadecathlon.com/
- http://grantmuller.com/projects/game-of-life/
- http://www.conwaylife.com/wiki/Conways_Game_of_Life
- http://www.radicaleye.com/lifepage/
- http://www.ibiblio.org/lifepatterns/

Further reading I

Dave Burraston, Ernest Edmonds, Dan Livingstone, and Eduardo Reck Miranda, *Cellular automata in MIDI based computer music*, ICMC, 2004.

Martin Gardner, *Mathematical games: The fantastic combinations of John Conway's new solitaire game "life"*, Scientific American **223** (1970), 120–123.

Deborah Gordon, *Ants at work: How an insect society is organized*, The Free Press, 1999.

P. Rendell, *A universal Turing machine in Conway's Game of Life*, International Conference on High Performance Computing Simulation, July 2011, pp. 764–772.

Stephen Wolfram, *A new kind of science*, Wolfram Media, 2002.