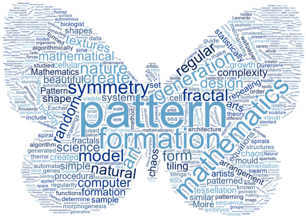
Advanced Mathematical Perspectives 1

Lecture 17: Reaction-Diffusion Systems



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SUB CRUCE LUMEN

ACEMS

AUSTRALIAN RESEARCH COUNCIL CENTRE OF EXCELLENCE FOR MATHEMATICAL AND STATISTICAL FRONTIERS

The profound study of nature is the most fertile source of mathematical discoveries.

Joseph Fourier (1768-1830)

'I'll take spots, then,' said the Leopard; 'but don't make 'em too vulgar-big. I wouldn't look like Giraffe–not for ever so.'

How the leopard got his spots, Just So Stories, Rudyard Kipling

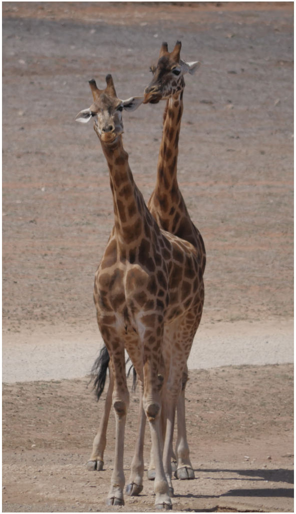
Section 1

Symmetry Breaking

Real Patterns

- Real patterns have periodicity
  - periodic patterns aren't perfectly symmetrical
  - *e.g.*, they have n-fold rotational symmetry, not arbitrary rotational symmetry
- Real patterns have some broken symmetries

Real Patterns



Skin/hair patterns

- Patterns on mammals are usually from coloured hair
  - colour from pigments: melanin (eumelanin and phaeomelanin)
  - pigment from special cells: melanocytes
- Pigments produced by melanocytes depend on presence/absence of activator/inhibitor chemicals

Buridan's Ass

Ass = Donkey

- Donkey is placed *exactly* between two precisely *equal* stacks of hay.
- The donkey will go to the best or closest, but they are the same.
- So it starves to death because it can't decide which to go to.

Named after 14th century philosopher Jean Buridan, but idea goes back at least to Aristotle 350 BC.

Ex nihilo nihil fit: Nothing comes out of nothing

The idea underlying Buridan's Ass

- Take a system that has a particular symmetry
- Assume the laws of physics are symmetric, and that the system evolves under these laws
- The system cannot "lose" the symmetry
- But there are many cases that seem to contradict this.

Symmetry breaking in nature

- Most higher-life forms on our planet starts as a single *spherical* cell, but end up with only with bilateral symmetry
- Even on bilaterally symmetric animals, we see patterns that are not symmetric, *e.g.*, giraffe spots
- A flower starts as an (almost) cylindrical stem, but then creates petals, with only discrete rotational symmetry.

Primary question: how can symmetry break?

Secondary question: when the symmetry breaks, why does it do so in such a controlled way, *e.g.*, why are there usually the same number of petals?

Section 2

Reaction-Diffusion Systems

We've seen *diffusion* and *reaction* — now we put them together

- 2D surface covered with two or more reagents, and $u_i(t, x, y)$ is the concentration of
  - reagent i
  - at time t
  - in location (x, y)
- Mixing only by diffusion
- Reactions are only local, *i.e.,* they depend on the concentrations at a point, not anywhere else

Individually, these are fairly simple, but together they can generate very interesting behaviour

Reaction-diffusion equation

$$\begin{aligned}\frac{\partial u_1}{\partial t} &= r_1(u_1, u_2) + \alpha_1 \nabla^2 u_1 \\ \frac{\partial u_2}{\partial t} &= r_2(u_1, u_2) + \alpha_2 \nabla^2 u_2\end{aligned}$$

- the terms $r_i(u_1, u_2)$ are reactions
  - implicitly they are dependent only on local concentrations
  - if $r_i = 0$ then this is just a diffusion system
- the terms $\alpha_i \nabla^2 u_i$ are diffusion terms
  - they are independent of reactions
  - if $\alpha_i = 0$ then it is just a reaction system

Examples: [Tur52]

Turing's first example of *morphogenesis*

- Assume we have a set of N cells arranged in a circle
- Discrete space, 1D version of the above

Turing analysed the *stability* of the *linearised* version of the system and showed that adding diffusion could make a stable reaction system into an unstable system.

- this is counter-intuitive because we usually think of diffusion (smoothing) as increasing stability
- result is demonstrated by finding the solution in terms of overlapping sinusoids
  - for certain parameters we get solutions in terms of sinusoids, so circular symmetry is broken in favour of n-fold rotational symmetry
  - *e.g.*, petal formation on a flower stem

Section 3

Pattern Formation and Instability

Schnakenberg system

$$\frac{\partial u_1}{\partial t} = \gamma\left(a - u_1 + u_1^2 u_2\right) + \alpha_1 \nabla^2 u_1$$
$$\frac{\partial u_2}{\partial t} = \gamma\left(b - u_1^2 u_2\right) + \alpha_2 \nabla^2 u_2$$

- u_1 is an *activator*
  - autocatalytic, *i.e.*, stimulates production of itself
  - slow diffusion (so short-range effect)
  - triggers “colour” in pattern
- u_2 is an *inhibitor*
  - reduces production of u_1 (and itself)
  - fast diffusion (long-range effect)

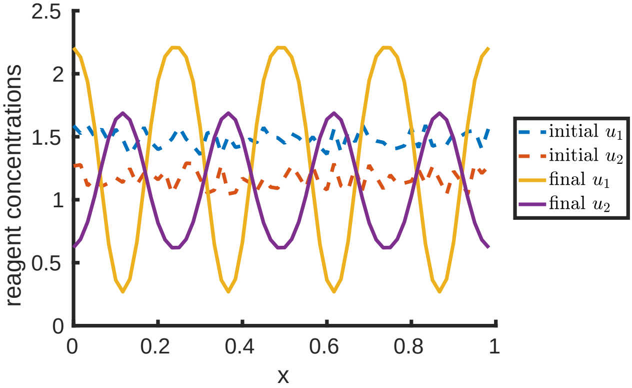
$$\alpha_2 > \alpha_1$$

- Start with noise around equilibrium

$$u_1^* = a + b, \quad u_2^* = \frac{b}{(a+b)^2}$$

Example: 1D

$\alpha_1 = 1.0, \alpha_2 = 4.8, \gamma = 1000, a = -0.55, b = 1.9$



- Note periodic pattern, despite noise input
  - shape of pattern isn't affected by noise (only start point)

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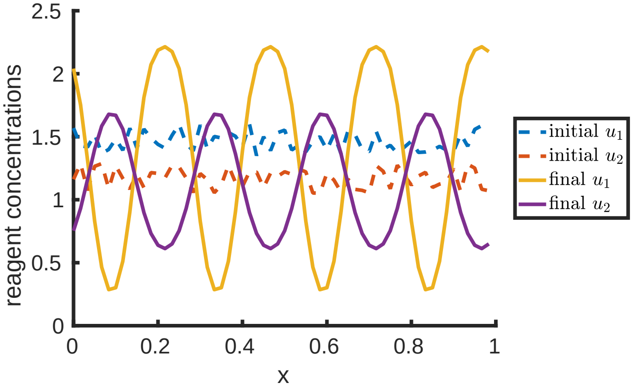

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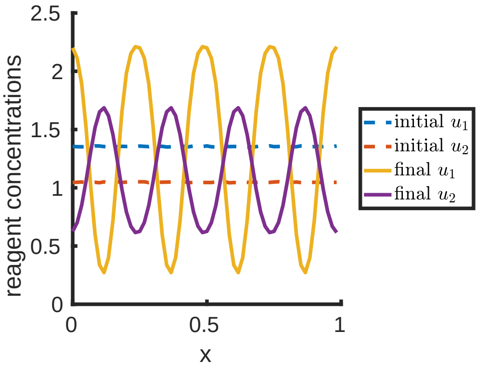

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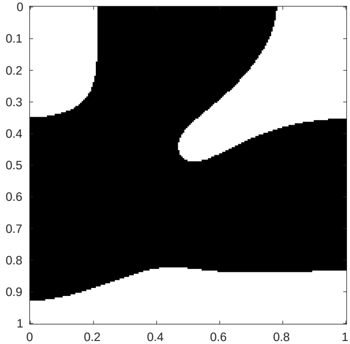

- Output has been thresholded to highlight it
- Made up of several periodic functions in different directions, so looks almost random
- Size matters: long thin grid results in stripes instead of spots

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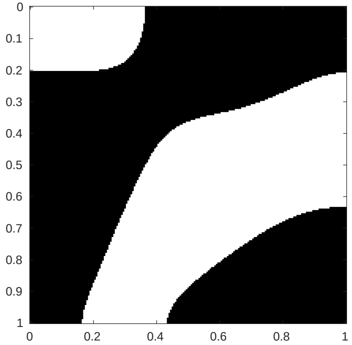

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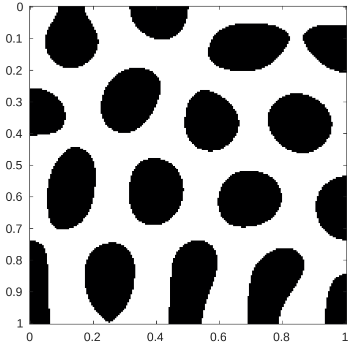

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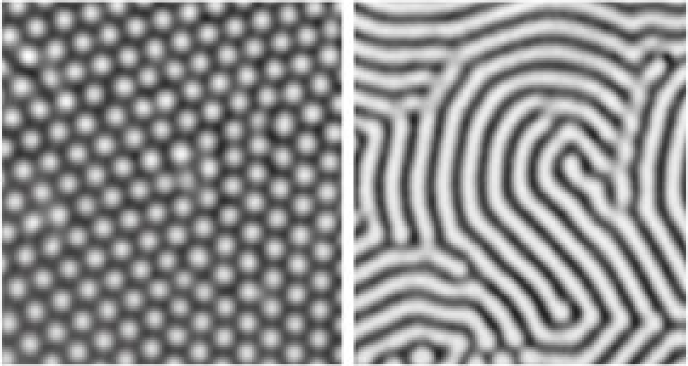
Example: other examples



https://www.chemistryworld.com/feature/turing-patterns/4991.article

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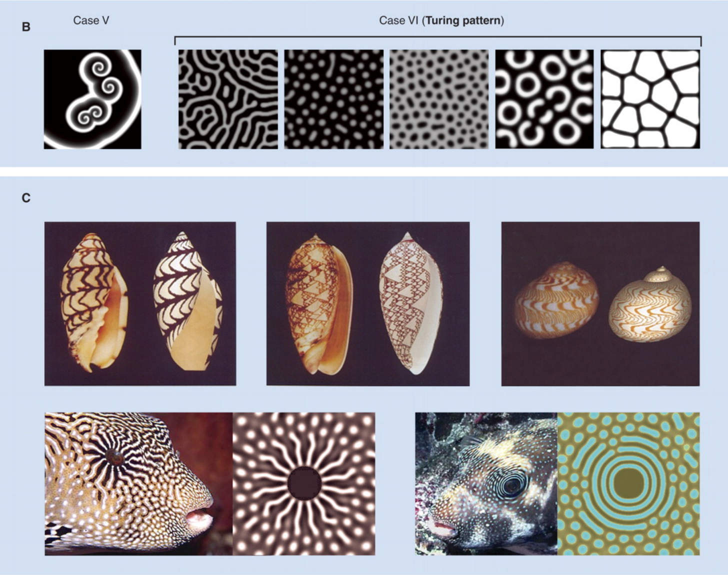
Examp



[KM10]

Takeaways

- Instability can lead to minuscule natural variations (noise) being enhanced, leading to apparent symmetry breaking
- The really interesting thing is that this can happen in a controlled way such that the resulting pattern is almost independent of the input noise
- The models above make detailed assumptions about processes, that might not be real, but the underlying idea is very deep

Section 4

Extras

Scale

- Scale is important here
  - determines the relative size of parameters
  - patterns form at some stage in foetal development, and size/shape of foetus at that point is important to eventual patterns

Links

- https://www.theguardian.com/science/punctuated-equilibrium/2010/oct/20/6
- https://www.popmath.org.uk/rpamaths/rpampages/leopard.html
- https://mosaicscience.com/story/how-zebra-got-its-stripes-alan-turing/
- https://thatsmaths.com/2013/04/25/spots-and-stripes/
- https://naiadseye.wordpress.com/2015/08/13/how-sea-shell-patterns-look-the-way-they-do/
- http://homepage.univie.ac.at/marie-therese.wolfram/teaching.html
- https://www.chemistryworld.com/feature/turing-patterns/4991.article

Further reading I

Shigeru Kondo and Takashi Miura, *Reaction-diffusion model as a framework for understanding biological pattern formation*, Science **329** (2010), no. 5999, 1616–1620.

Boyce Tsang, *Patterns in reaction diffusion system*, 2011, guava.physics.uiuc.edu/~nigel/courses/569/Essays_Fall2011/Files/tsang.pdf.

A.M. Turing, *The chemical basis for morphogenesis*, Philosophical Transactions of the Royal Society of London B **237** (1952), 37–72, www.dna.caltech.edu/courses/cs191/paperscs191/turing.pdf.