

Advanced Mathematical Perspectives 1

Lecture 7: Periodicity and superposition of sines



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MATHEMATICAL AND STATISTICAL FRONTIERS

Section 1

Sinusoidal Patterns

And last of all we have the secondary forms of crystals bursting in upon us, and sparkling in the rigidity of mathematical necessity and telling us, neither of harmony of design, usefulness or moral significance, nothing but spherical trigonometry and Napier's analogies. It is because we have blindly excluded the lessons of these angular bodies from the domain of human knowledge that we are still in doubt about the great doctrine that the only laws of matter are those which our minds must fabricate, and the only laws of mind are fabricated for it by matter.

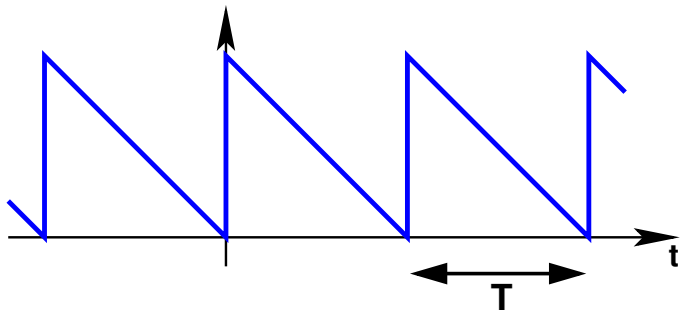
James Clerk Maxwell

Periodicity

- One of the symmetries we saw before was translational, which, leads to *periodicity*
 - ▶ regular tilings were all periodic
- We're going to look into this a great deal more today
- Let's start in 1D, and then move to 2D
- We'll start with a formal definition, and then move onward

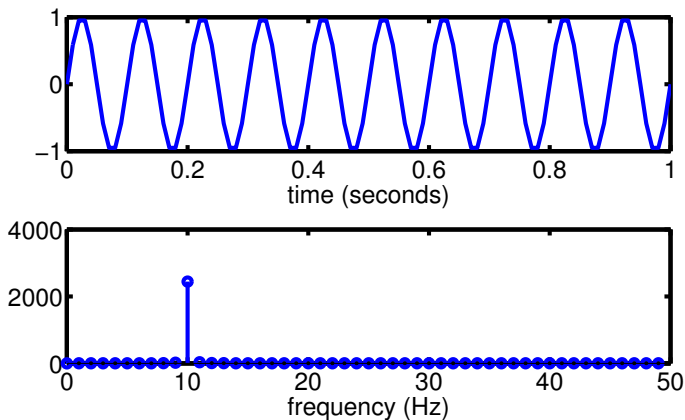
Periodicity

$$x(t + nT) = x(t) \text{ for any } n = 1, 2, \dots$$



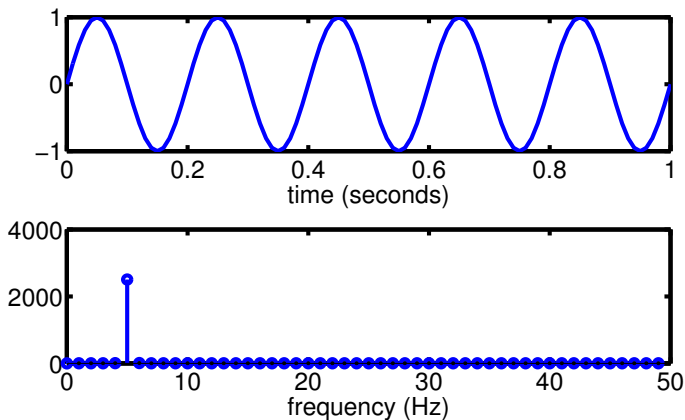
- Period = T
(the minimal value for which the above is true)
- Frequency $f = 1/T$
(if T is a measure of time, frequency is measured in Hz).

Sinusoids are the most common example of periodic signals



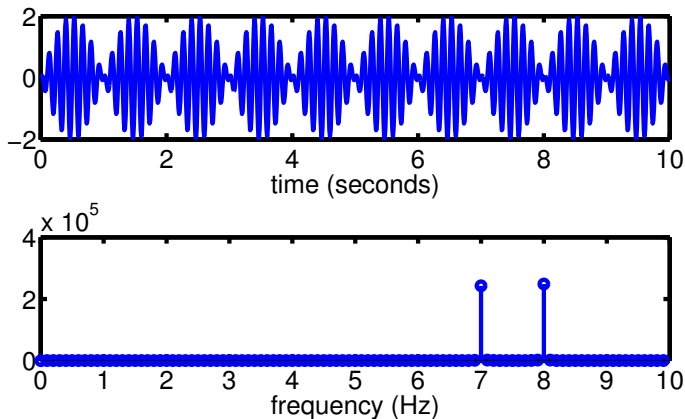
The figure shows a sine wave $y = \sin(2\pi ft)$, and its *power spectrum*.

Sinusoids are the most common example of periodic signals



The figure shows a sine wave $y = \sin(2\pi ft)$, and its *power spectrum*.

Two sine waves combine to create *beats*



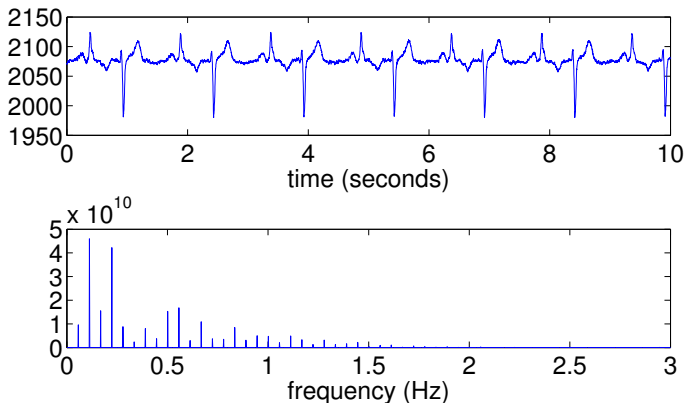
Sum-to-product trig identity

$$y = \sin(2\pi f_1 t) + \sin(2\pi f_2 t) = 2 \sin\left(\frac{2\pi(f_2 + f_1)t}{2}\right) \cos\left(\frac{2\pi(f_2 - f_1)t}{2}\right)$$

Periodicity and Sinusoids

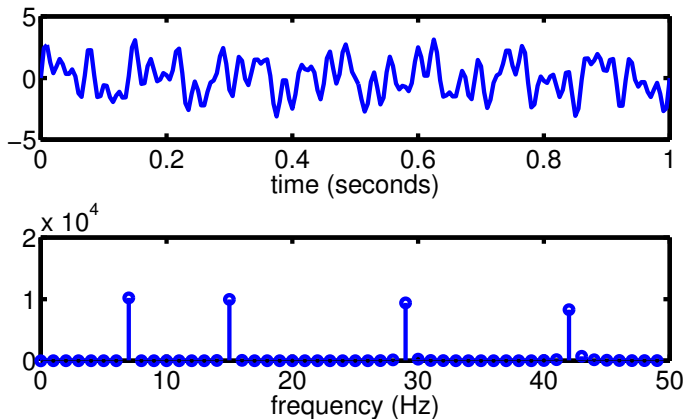
- We can model *any* periodic function as a (possibly infinite) sum of sinusoids
- At the moment, you have to take this on faith, but I will provide a method for getting the right sine waves later on
- What's more interesting, is that we can also generate apparently random signals from sine waves

Example: ECG



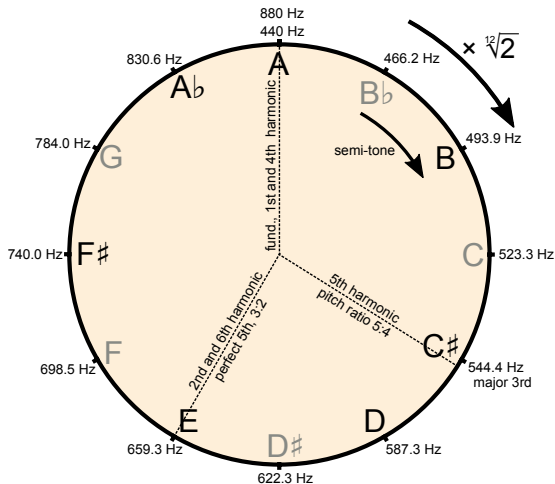
How does the machine that goes “beep” in a hospital work? Its looking for the period of your heartbeat, looking at a signal like this. The figure shows the (slightly) more complex set of sine wave frequencies I need to model this signal.

Example: almost random



This shows the sum of only *four* sine waves to generate a signal that to all appearances seems random. It isn't, but its period is much longer than the segment shown.

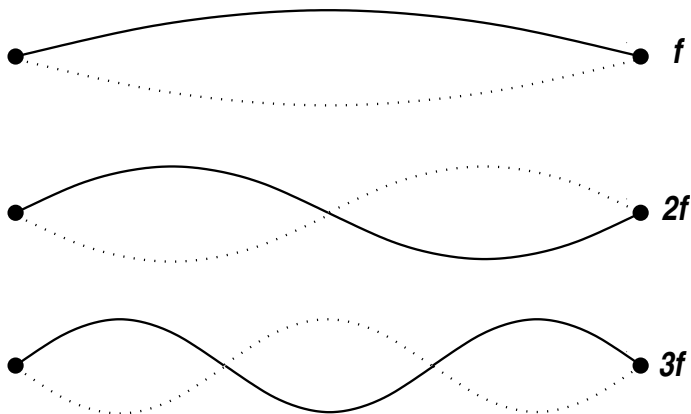
Equal temperament scale in music



The scale is *logarithmic* and *modular* (mod 12)

Patterns in sounds: Harmonics

Real instruments don't generate pure sin waves



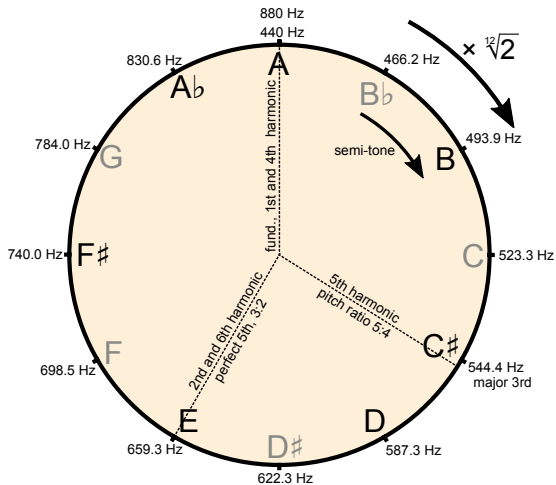
Vibrational resonances at fundamental frequency f and at $2f, 3f, \dots$

We hear a mix of these *harmonics*

Harmonics of A (440 Hz)




Harmonic	Frequency	Normalized	Note name	how close
1 (fundamental)	440Hz	440Hz	A	100%
2	880Hz	440Hz	A	100%
3	1320Hz	660Hz	E	100%
4	1760Hz	440Hz	A	100%
5	2200Hz	550Hz	C#	99%
6	2640Hz	660Hz	E	100%
7	3080Hz	770Hz	G	98%
8	3520Hz	440Hz	A	100%
9	3960Hz	495Hz	B	100%
10	4400Hz	550Hz	C#	99%
11	4840Hz	605Hz	D	103%
12	5280Hz	660Hz	E	100%
13	5720Hz	715Hz	F#	97%
14	6160Hz	770Hz	G	98%
15	6600Hz	825Hz	G#	99%

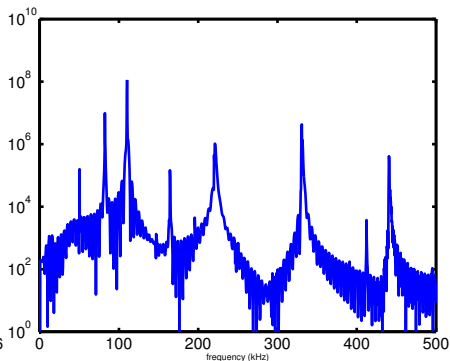
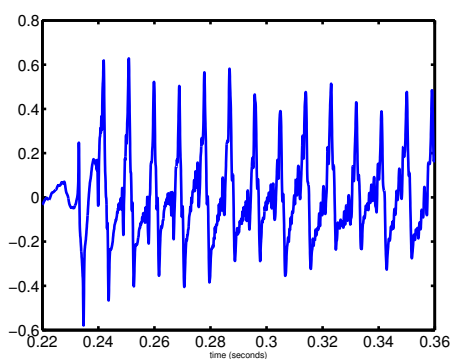
Harmonics and scales



Tone/Timbre of instrument

Tone/Timbre of instrument is in part determined by proportion of different harmonics.

- Cosine, 110 Hz , Cosine, 440 Hz 
- Guitar, A 



Section 2

An aside: pseudo-random numbers

An aside: pseudo-random numbers

- Computers are *deterministic*

- ▶ given a certain state, and input, they always produce the same output
- ▶ maybe doesn't seem that way, but remember a computer with 4 GB of RAM can have

$$2^{4,000,000,000}$$

states, so they can be somewhat hard to anticipate

- ▶ the whole universe has only $\sim 10^{80} \simeq 2^{270}$ fundamental particles
<https://www.physicsoftheuniverse.com/numbers.html>

- But sometimes we would like *random* numbers

- ▶ computer games
- ▶ cryptography
- ▶ simulations

An aside: pseudo-random numbers

- Computers generate *pseudo-random* numbers
- Simple algorithm is *linear congruential generator*

$$X_{n+1} = (aX_n + c) \pmod{m}$$

where we have multiplier a , increment c , and the modulus m

- Essentially this generates a sequence with a *very* long period
 - ▶ period can't be longer than m , but can be of that order
 - ▶ example [PFTV92, p.284]: $a = 1664525, c = 1013904223, m = 2^{32}$
- Note, there are better approaches now, but we won't go into them in detail
 - ▶ [PFTV92, p.284] call this a quick and dirty approach
 - ▶ Mersenne twister has period $2^{19937} - 1$

Section 3

Spirograph

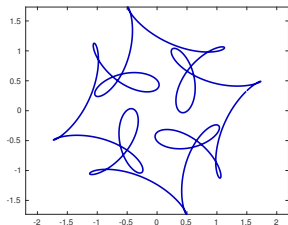
Parametric curves

Another way to extend to 2D is through parametric curves: *i.e.*, we express the (x, y) coordinates as a function of some *parameter* t .

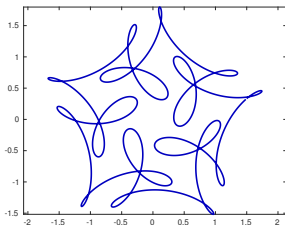
Example:

$$x(t) = \cos(t) + \cos(n_1 t)/2 + \sin(n_2 t)/3$$

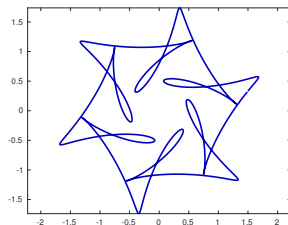
$$y(t) = \sin(t) + \sin(n_1 t)/2 + \cos(n_2 t)/3$$



$$n_1 = 5, n_2 = 11$$



$$n_1 = 6, n_2 = 14$$



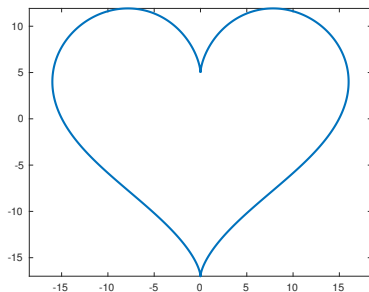
$$n_1 = 7, n_2 = 11$$

Non-linear parametric equations

We can do even more with non-linear curves

$$x(t) = 16 \sin^3(t),$$

$$y(t) = 13 \cos(t) - 5 \cos(2t) - 2 \cos(3t) - \cos(4t).$$



Polar coordinates

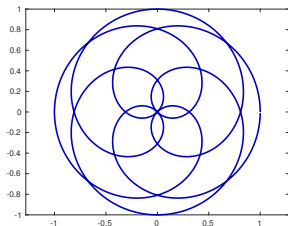
Yet another way is to express a curve in polar coordinates (θ, r)

$$x = r \cos \theta,$$

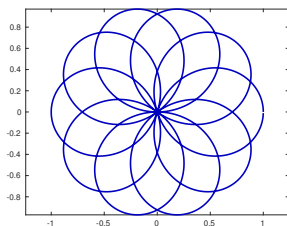
$$y = r \sin \theta.$$

Example: a *rose* or *rhodonea* curve

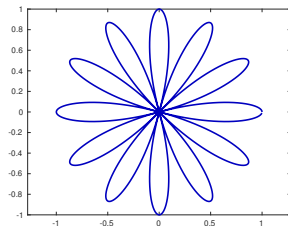
$$r = \cos\left(\frac{n}{d}\theta\right).$$



$$n = 2, d = 5$$



$$n = 5, d = 4$$



$$n = 6, d = 1$$

Other topics

- Non-linear excitation can cause *Faraday waves*, which can exhibit interesting patterns.
- Lissajous Curves and the ABC logo

Activity

Generate some sinusoidal curves

Further reading I



William H. Press, Brian P. Flannery, Saul A. Teukolsky, and William T. Vetterling, *Numerical recipes in c: The art of scientific computing*, 2nd ed., Cambridge University Press, 1988-92, Available at <http://www.nr.com/>.